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Free Vibration of Laminated Plates

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ABSTRACT

This paper addresses free vibration of laminated plates using finite element method with the intention of studying the effects of certain parameters on the natural frequency. First – order shear deformation theory is employed in the analysis. A computer program is written in matlab. The factors which were studied included lamination sequence, material properties, aspect ratio, length/ thickness ratio, length/ width ratio, and boundary conditions. It is found that, the length/ width ratio has the most serious effects on the natural frequency.

KEYWORDS: *Free vibration, laminated plates, first order shear deformation theory, Matlab, finite element method.*

1. INTRODUCTION

Composites are man – made materials consisting of fiber embedded in matrix. They are acknowledged for their superior properties compared with traditional structural materials such as high strength – to – weight ratio, and high stiffness. The fiber is the main load carrier whereas the matrix supports the fiber and transfers the external load to the fiber through the fiber – matrix interface shear stresses at the fiber tips. Composites are manufactured in the form of thin sheets, and when a number of such sheets are bonded together they form a structural element called a laminate.

A laminated plate is an example of a structural element made of unidirectional plies (i.e. sheets) stacked together with appropriate orientation to achieve design requirements.

This paper studies free vibration of rectangular laminated plates. The knowledge of the few lower natural frequencies is important in order to avoid failure of the structure when it is subjected to forced vibration. Failure could be immediate due to large amplitudes of motion, or short life failure due to fatigue. The paper investigates the effects of some parameters on the natural frequencies such as boundary conditions, the lamination sequence, length/ width ratio, length/ thickness ratio, and the properties of the composite material.

It is well known that shear deformation is more pronounced in composites than in metals due to their low shear modulus to in – plane elastic modulus. This led to the evolution of a number of mathematical models to simulate the static and dynamic behavior of a laminate. These models include the classical theory that ignores shear deformation and rotary inertia [1], first – order shear deformation theory [2] that assumes constant shear deformation across the laminate's thickness, and a variety of higher – order shear deformation theories [3] that claim better representation of the deformed cross – sections i.e. parabolic shear stress distribution.

The majority of the mathematical models replace the laminate with an equivalent single layer, then a displacement field is assumed through thickness, and the equivalent properties of the laminate is then expressed in term of the properties of the individual plies.

Free vibration of laminated plates has been subject of research for the last four decades. A large amount of literature has been published. Closed form solution is available for laminates with certain lamination sequence and boundary conditions. Numerical techniques are employed in the study of laminate behavior and in particular finite element method. The reader is referred to a few of the published works in Refs. [4] – [18]. Despite the large amount of available literature, the study of free vibration remains a very interesting subject. The driving force for further investigations is, as ever, achieving higher accuracy and more efficient computation techniques.

2. MATHEMATICAL FORMULATION

Consider a plate of length a, breadth b and depth h, consisting of n plies with varying thickness, orientation, and properties as shown in Figure 2.1.

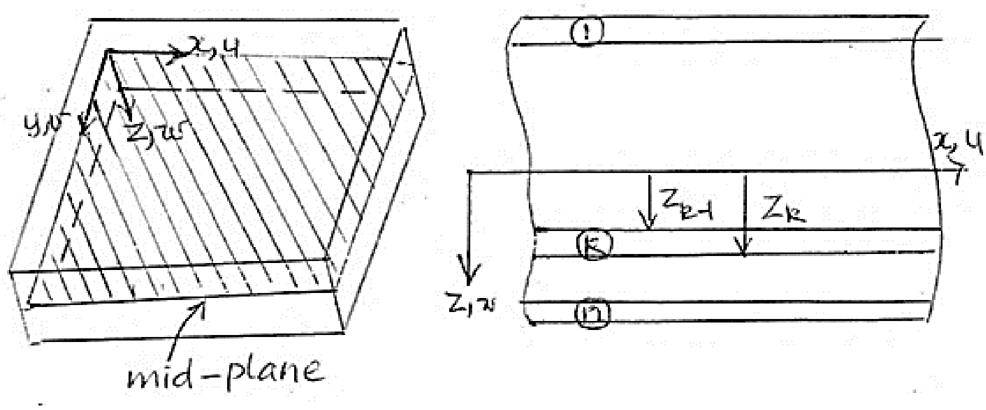

Figure 2.1

The displacement field according to first – order shear deformation theory is:

$$U(x, y, z, t) = u(x, y, t) + z\phi(x, y, t)$$

$$V(x, y, z, t) = v(x, y, t) + z\psi(x, y, t)$$

$$W(x, y, z, t) = w(x, y, t) \qquad (1)$$

Where u, v, and w are the mid – plane displacements in the x, y, and $z-$ directions respectively. ψ and ϕ are the rotations about x and $y-$ axes respectively. t is time.

The strain – displacement relations are:

$$\epsilon_1 = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x}$$

$$\epsilon_2 = \frac{\partial v}{\partial y} + z\frac{\partial \psi}{\partial y}$$

$$\epsilon_4 = \frac{\partial w}{\partial y} + \psi$$

$$\epsilon_5 = \frac{\partial w}{\partial x} + \phi$$

$$\epsilon_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + z\left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\right) \qquad (2)$$

The subscripts are in accordance with the three – dimensional elasticity formulation.

We choose the 8 – noded element shown in Figure 2.2 below.

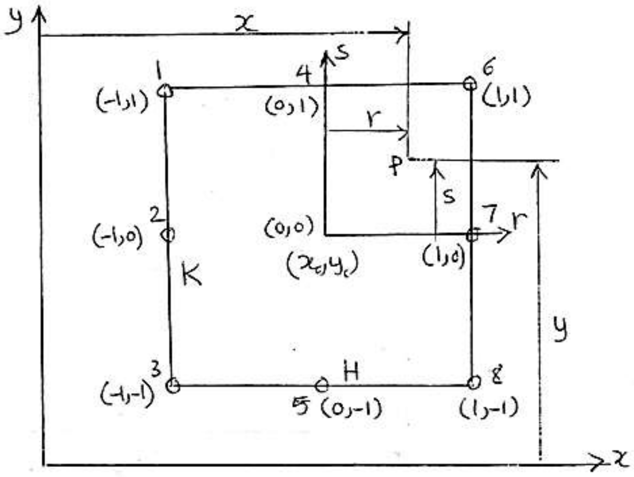

Figure 2.2

The displacements can be expressed in terms of shape functions N_i as follows:

$$\alpha = N_i \, \alpha_i$$

Where α stands for u, v, w, ϕ, ψ and α_i are the corresponding nodal values of these variables. The shape function can be expressed in terms of local coordinates as follows:

$$N_i = a_{i,1} + a_{i,2}r + a_{i,3}s + a_{i,4}rs + a_{i,5}rs^2 + a_{i,6}r^2s + a_{i,7}s^2 + a_{i,8}r^2 \quad (3)$$

Where $i = 1,2, \dots, 8$

The relations between the local coordinates (r, s) and the global coordinates (x, y) are: $x = x_o + \frac{H}{2}r$ and $y = y_o + \frac{K}{2}s$.

The coefficients are given in Appendix (A).

The strains can be written in matrix form as follows:

$$\epsilon = Ba^e \qquad (4)$$

where,

$$B = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & 0 z\frac{\partial N_i}{\partial r} & 0 \\ 0 & \frac{\partial N_i}{\partial s} & 00 & z\frac{\partial N_i}{\partial s} \\ 0 & 0 & \frac{\partial N_i}{\partial s}0 & N_i \\ 0 & 0 & \frac{\partial N_i}{\partial r}N_i & 0 \\ \frac{\partial N_i}{\partial s} & \frac{\partial N_i}{\partial r} & 0z\frac{\partial N_i}{\partial s} & z\frac{\partial N_i}{\partial r} \end{bmatrix} \quad i = 1, 8$$

The vector of nodal displacements is expressed as follows:

$$a^e = [u_i v_i w_i \phi_i \psi_i] \qquad i = 1, 8$$

The stress – strain relation:

$$\sigma = C\,\epsilon \qquad (5)$$

Where $\sigma = [\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5]^T$ and C is the matrix of the transformed properties, given for an orthotropic laminate as:

$$C = \begin{bmatrix} c_{11} & c_{12} & 0\,0 & c_{16} \\ c_{12} & c_{22} & 00 & c_{26} \\ 0 & 0 & c_{44}c_{45} & 0 \\ 0 & 0 & c_{45}c_{55} & 0 \\ c_{16} & c_{26} & 0\ \ 0 & c_{66} \end{bmatrix} \qquad (6)$$

C_{ij} are given in Appendix (B).

The strain energy,

$$U_s = \frac{1}{2}\int_V \epsilon^T \sigma dV \qquad (7)$$

When equation (4) and equation (5) are substituted in equation (7):

$$U_s = \frac{1}{2}\int_V (Ba^e)^T C(Ba^e)\, dV$$

$$or \qquad U_s = \frac{1}{2} a^{e^T} \int_V B^T C\, Ba^e\, dV$$

Which can be written in the form:

$$U_s = \frac{1}{2} a^{e^T} K^e a^e$$

Where K^e is the element stiffness matrix.

$$K^e = \int_V B^T C\ B\ dV$$

Work done by inertia forces is expressed as shown below:

$$T = \frac{1}{2}\int_V \rho\left[\frac{\partial^2 U}{\partial t^2}U + \frac{\partial^2 V}{\partial t^2}V + \frac{\partial^2 W}{\partial t^2}W\right]dV$$

When introduce equation (1) the work done will be as follows:

$$T = \frac{1}{2}\int \rho\left[(u+z\phi)\frac{\partial^2}{\partial t^2}(u+z\phi) + (v+z\psi)\frac{\partial^2}{\partial t^2}(v+z\psi) + w\frac{\partial^2 w}{\partial t^2}\right]$$

Where ρ is the mass density.

It is assumed that motion due to vibration is harmonic.

$$i.e. \qquad \frac{\partial^2 \alpha}{\partial t^2} = -w^2\alpha$$

Where α stands for u, v, w, ϕ, ψ ; and ω is the natural frequency. Hence,

$$T = -\frac{1}{2}\rho\omega^2\int_V [u \ \ v \ \ w \ \ \phi \ \ \psi] Z \begin{bmatrix} u \\ v \\ w \\ \phi \\ \psi \end{bmatrix} dV$$

Introducing the shape functions, we can express the work done by inertia forces as:

$$T = -\frac{1}{2}\rho\omega^2 a^{e^T}\int_V N^T Z a^e dV$$

where,

$$N = \begin{bmatrix} N_i & 0 & 00 & 0 \\ 0 & N_i & 00 & 0 \\ 0 & 0 & N_i0 & 0 \\ 0 & 0 & 0N_i & 0 \\ 0 & 0 & 00 & N_i \end{bmatrix} \qquad i = 1,8$$

And,

$$Z = \begin{bmatrix} 1 & 0 & 0Z & 0 \\ 0 & 1 & 00 & Z \\ 0 & 0 & 10 & 0 \\ Z & 0 & 0Z^2 & 0 \\ 0 & Z & 00 & Z^2 \end{bmatrix}$$

$$\therefore \quad T = -\frac{1}{2}a^{e^T}w^2 M^e a^e$$

where M^e is the element mass matrix.

$$M^e = \int_V \rho N^T Z N dV$$

In the absence of damping and external loads, the total energy can be expressed as:

$$U_s + T = 0$$

$$i.e.\quad \frac{1}{2}a^{e^T}K^e a^e - \frac{1}{2}a^{e^T}\omega^2 M^e a^e = 0$$

$$i.e.\quad [K^e - \omega^2 M^e]a^e = 0$$

This can be expressed globally as:

$$[K - \omega^2 M]a = 0$$

where,

$$K = \sum K^e\ ,\ \ M = \sum M^e\ ,\ \ a = \sum a^e$$

The elements of the stiffness and mass matrices are given in Appendix (C).
Computations are carried using the non – dimensional quantities given in Appendix (D).

3. RESULTS AND DISCUSSION

Finite element method has been formulated. Integrations encountered in the analysis have been done by hand. A matlab program was then compiled and subjected to verification. New results have been produced. However, only few samples of the computed results are presented using 10 by 10 finite element mesh. Three types of material properties and three types of plate boundary conditions are employed in the analyses, and these are as follows:
Material properties:
Material (1): $E_1 = 10E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, v_{12} = 0.25$
Material (2): $E_1 = 25E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, v_{12} = 0.25$
Material (3): $E_1 = 40E_2, G_{12} = G_{13} = 0.6E_2, G_{23} = 0.5E_2, v_{12} = 0.25$
Boundary conditions:
$SS1$: $at\ x = 0$ and $x = a$: $v = 0, w = 0, thi = 0.$
$at\ y = 0$ and $y = b$: $u = 0, w = 0, phi = 0.$
$SS2$: $at\ x = 0$ and $x = a$: $u = 0, w = 0, thi = 0.$
$at\ y = 0$ and $y = b$: $v = 0, w = 0, phi = 0.$
CC: $at\ x = 0, x = a, y = 0, y = b: u = 0, v = 0, w = 0, phi = 0, thi = 0.$

FF: All edges of plate are free.

To verify the accuracy of the computed results, comparison is made with Ref. [15]. First, Table (3.1) lists the non – dimensional fundamental frequency of a 2 – layer and 8 – layer anti – symmetric cross – ply material (1) laminate with boundary conditions SS1. The results are given for two aspect ratios (a/b) 1 and 3, and three values of length/ depth ratios. An eye scan of the results reveals an excellent comparison. Second, Table (3.2) lists the non – dimensional fundamental frequency of a 2 – layer and 8 – layer anti – symmetric angle – ply square laminate with boundary conditions SS2. The results are given for two sets of material properties: material (2) and material (3). Again, there is a good agreement between the two sets of results. The differences between the two sets are slightly bigger for the 2 – layer plate with the higher modular ratio because the behavior of such laminates is sensitive to coupling between extension and bending.

New results are generated for 3 – layer symmetric cross – ply 0/90/0 material (1) laminates. The plates analyzed are either square (a=b), or rectangular (a=2b). The width/thickness ratio is varied from 5 to 80 to cover most of the practical plate's dimensions. Two extreme boundary conditions are considered i.e. all edges are either free (FF) or built – in (CC). The first five lower non – dimensional frequencies are given in Appendix (E): Table (E.1) – Table (E.6).

The results given in the tables enable one to highlight the following points:

1. The built – in plate (CC) have higher values of frequency compared with those of the free plate (FF) ignoring the rigid body motion. For example, it can be shown from the results given in Table (E.1) and Table (E.3), that the non – dimensional frequency of a built – in plate when a/h=5 is greater than the frequency of the same plate when it is free by 152.8%. When a/h=80, the percentage is 303.6%. Hence it can be concluded that the effect of the support becomes more and more noticeable as the length/ thickness ratio increases.

2. As a plate becomes thinner (i.e. a/h becomes larger), the non – dimensional frequency increases, and ultimately tends to assume a stationary value (see Figure 3.1 and Figure 3.2). However, the actual (dimensional) frequency tends to decrease rapidly as the plate becomes thinner because the actual frequency is inversely proportional to the square of the length/ thickness ratio. For example, we find from Table (E.1), the fundamental frequency, of a square laminate is 1.2479 when a/ h=5, and 1.4892 when a/ h=80, which means an increase of about 19.3%. Instead of the apparent increase of the non – dimensional frequency, it can be shown that the actual frequency of the thin laminate has dropped by more than 99.5% as compared with the thick laminate frequency. Hence we conclude that the length/ thickness ratio has a profound and dramatic effect on the actual frequency of a plate, and hence that parameter should be closely observed when analyzing laminated plates.

3. As the width of plate decreases, the non – dimensional frequency tends to rise, while the actual frequency decreases rapidly. For example, the fundamental frequency of laminates with a=b and a/h=5 from Table (E.1) is 1.2479, and for a=2b and a/h=5 from Table (E.2) is 2.1989. That means an increase of 76.2%. However, the actual frequency can be shown to have dropped by more than 55.9%. Hence we can say that the aspect ratio of a plate (a/b) has noticeable effect on the actual frequency of a plate but that effect does not amount to the effect of length/ depth ratio.

4. As the modular ratio, E_1/E_2, increases, the non – dimensional frequency decreases for the same plate with the same boundary conditions. To prove that consider the results given in Table (E.1) and Table (E.5). The two sets of results are for two similar plates, but two different materials. The modular ratio E_1/E_2 is 10 for the results in Table (E.1), and 40 for those results in Table (E.5). When a/h=5, the fundamental frequencies, are 1.2479 and 0.6257 for modular ratio of 10 and 40 respectively. That is to say the non – dimensional frequency has dropped by 49.9% as the modular ratio increases from 10 to 40. If we assume that the transverse modulus E_2 is the same in both cases, then we can show that the actual frequency, rather than dropping, has increased slightly by about 2.8%. That is to say the effect of the modular ratio, or the properties of the composite material, in general, has the least pronounced effect on the frequency.

5. A great amount of results have been produced for different lamination lap – ups, boundary conditions, and material properties which, of course, cannot be accommodated in the limited space permitted by the publisher. However, there is one single case worth mentioning. That case concerns an 8 – layer cross – ply $(0/90)_4$ laminate, material (3), boundary conditions SS1, aspect ratio a/b=3, length/ depth ratio 10. The present study gives the non – dimensional fundamental frequency $\varpi = \omega b^2(p/E_2h^2)^{1/2} = 7.405$, and the second frequency 10.269 . However, Reddy [15] gives the fundamental frequency as 10.269 which corresponds exactly to the second frequency according to the present work. Among lots of results produced, that one single frequency sounds odd, and hence deserves further consideration.

Table (3.1): Non – dimensional fundamental frequency of cross – ply rectangular plates SS1 boundary conditions, Material (1).

a/h		**a=b**		**a=3b**	
		0/90	$(\mathbf{0/90})_4$	**0/90**	$(\mathbf{0/90})_4$
10	present	7.454	9.450	4.751	6.319
	Ref. [15]	7.454	9.450	4.751	6.319
20	present	7.804	10.103	4.910	6.673
	Ref. [15]	7.802	10.102	4.908	6.592
100	present	7.943	10.362	4.976	6.813
	Ref. [15]	7.926	10.354	4.962	6.799

Table (3.2): Non – dimensional fundamental frequency of angle – ply square plates, boundary conditions SS2.

		Material (2)		**Material (3)**	
a/h		**-45/45**	$(\mathbf{-45/45})_4$	**-45/45**	$(\mathbf{-45/45})_4$
5	present	8.498	10.285	10.336	12.893
	Ref. [15]	8.498	10.285	10.799	12.893
10	present	10.896	15.388	13.044	19.290
	Ref. [15]	10.895	15.388	13.629	19.289
20	present	11.934	18.556	14.181	23.260
	Ref. [15]	11.933	18.555	14.815	23.259
100	present	12.352	20.095	14.631	25.189
	Ref. [15]	12.341	20.084	15.274	25.176

4. CONCLUSIONS

1. Finite element method incorporating first – order shear deformation theory is used to study rectangular laminated plates in order to assess the effects of the plate parameters on its natural frequencies.
2. Excellent agreements is found between the present work and similar ones found in literature which encouraged the authors to produce extra vibration results with the aim to study the effects of lamination sequence, material properties, aspect ratio, length/ depth ratio, and boundary conditions on the frequency.
3. As expected, it is found that all the parameters mentioned above have influence on the frequency but to different extent. The greatest effect is caused by the length/ width ratio, and therefore it should be carefully considered when analyzing laminated plates. The least effect is due to the modular ratio or the properties of the composite material.

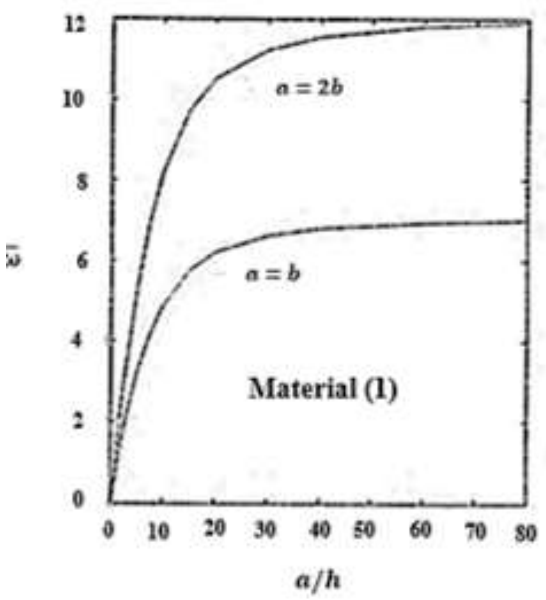

Figure 3.1 Fundamental frequency against length/ thickness ratio for clamped plate

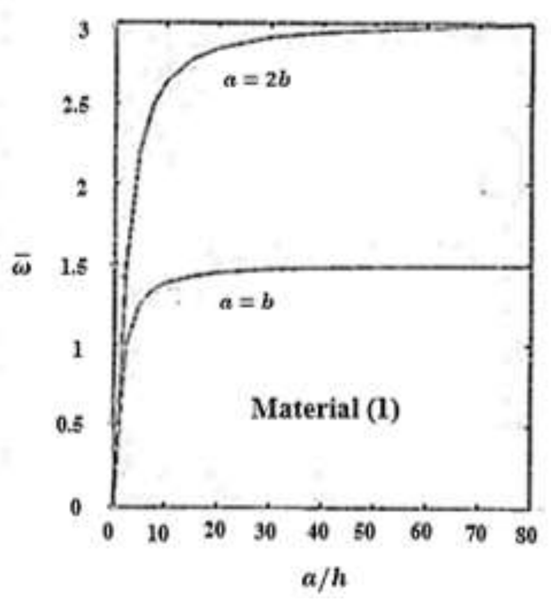

Figure 3.1 Fundamental frequency against length/ thickness ratio for free plate

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APPENDICES

Appendix (A)

a/4

	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$	$a_{i,7}$	$a_{i,8}$
N_1	-1	0	0	-1	-1	1	1	1
N_2	2	-2	0	0	2	0	-2	0
N_3	-1	0	0	1	-1	-1	1	1
N_4	2	0	2	0	0	-2	0	-2
N_5	2	0	-2	0	0	2	0	-2
N_6	-1	0	0	1	1	1	1	1
N_7	2	2	0	0	-2	0	-2	0
N_8	-1	0	0	-1	1	-1	1	1

Appendix (B)

$$C_{11} = C'_{11}c^4 + C'_{22}s^4 + 2(C'_{12} + 2C'_{66})s^2c^2$$

$$C_{12} = (C'_{11} + C'_{22} - 4C'_{66})s^2c^2 + C'_{12}(s^4 + c^4)$$

$$C_{22} = C'_{11}s^4 + C'_{22}c^4 + 2(C'_{12} + 2C'_{66})s^2c^2$$

$$C_{16} = (C'_{11} - C'_{12} - 2C'_{66})c^3s - (C'_{22} - C'_{12} - 2C'_{66})s^3c$$

$$C_{26} = (C'_{11} - C'_{12} - 2C'_{66})s^3c - (C'_{22} - C'_{12} - 2C'_{66})c^3s$$

$$C_{66} = (C'_{11} + C'_{22} - 2C'_{12} - 2C'_{66})s^2c^2 + C'_{66}(s^4 + c^4)$$

$$C_{44} = C'_{44}c^2 + C'_{55}s^2$$

$$C_{45} = (C'_{55} - C'_{44})sc$$

$$C_{55} = C'_{44}s^2 + C'_{55}c^2$$

where,

$$C'_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, C'_{11} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

$$C'_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \qquad C'_{44} = G_{23}$$

$$C'_{55} = G_{13}, \qquad C'_{66} = G_{12}$$

$s = \sin\theta$, $c = \cos\theta$

θ is the angle between the material direction and the laminate direction.

Appendix (C)

C.1 The element stiffness matrix:

The element stiffness matrix is a 40×40 symmetric matrix. In what follows $i, j = 0,1,2, \dots ,7$

$K_{1+5i,1+5j} = A_{11}q_1 + A_{16}(q_3 + q_4) + A_{66}q_2$

$K_{1+5i,2+5j} = A_{12}q_3 + A_{16}q_1 + A_{26}q_2 + A_{66}q_4$

$K_{1+5i,3+5j} = 0$

$K_{1+5i,4+5j} = B_{11}q_1 + B_{16}(q_3 + q_4) + B_{66}q_2$

$K_{1+5i,5+5j} = B_{12}q_3 + B_{16}q_1 + B_{26}q_2 + B_{66}q_4$

$K_{2+5i,1+5j} = A_{12}q_4 + A_{16}q_1 + A_{26}q_2 + A_{66}q_3$

$K_{2+5i,2+5j} = A_{22}q_2 + A_{26}(q_3 + q_4) + A_{66}q_1$

$K_{2+5i,3+5j} = 0$

$K_{2+5i,4+5j} = B_{12}q_4 + B_{16}q_1 + B_{26}q_2 + B_{66}q_3$

$K_{2+5i,5+5j} = B_{22}q_2 + B_{26}(q_3 + q_4) + B_{66}q_1$

$K_{3+5i,1+5j} = 0$

$K_{3+5i,2+5j} = 0$

$K_{3+5i,3+5j} = A_{44}q_2 + A_{45}(q_3 + q_4) + A_{55}q_1$

$K_{3+5i,4+5j} = A_{45}q_8 + A_{55}q_7$

$K_{3+5i,5+5j} = A_{44}q_8 + A_{45}q_7$

$K_{4+5i,1+5j} = B_{11}q_1 + B_{16}(q_3 + q_4) + B_{66}q_2$

$K_{4+5i,2+5j} = B_{12}q_3 + B_{16}q_1 + B_{26}q_2 + B_{66}q_4$

$K_{4+5i,3+5j} = \lambda^2 A_{45}q_6 + \lambda^2 A_{55}q_5$

$K_{4+5i,4+5j} = D_{11}q_1 + D_{16}(q_3 + q_4) + D_{66}q_2 + \lambda^2 A_{55}q_9$

$K_{4+5i,5+5j} = D_{12}q_3 + D_{16}q_1 + D_{26}q_2 + D_{66}q_4 + \lambda^2 A_{45}q_9$

$K_{5+5i,1+5j} = B_{12}q_4 + B_{16}q_1 + B_{26}q_2 + B_{66}q_3$

$$K_{5+5i,2+5j} = B_{22}q_2 + B_{26}(q_3 + q_4) + B_{66}q_1$$

$$K_{5+5i,3+5j} = \lambda^2 A_{44}q_6 + \lambda^2 A_{45}q_5$$

$$K_{5+5i,4+5j} = D_{12}q_4 + D_{16}q_1 + D_{26}q_2 + D_{66}q_3 + \lambda^2 A_{45}q_9$$

$$K_{5+5i,5+5j} = D_{22}q_2 + D_{26}(q_3 + q_4) + D_{66}q_1 + \lambda^2 A_{44}q_9$$

In the above expressions,

$$[A_{ij}, B_{ij}, D_{ij}] = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} C_{ij}[1, z, z^2]dz \qquad (i,j = 1,2,6)$$

$$A_{ij} = K_f \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} C_{ij}dz \qquad (i,j = 4,5)$$

Where K_f is shear correction factor taken as 5/6, and C_{ij} are defined in Appendix (A). The integrals $q_i(i = 1,2,\dots,9)$ are derived and given in the non – dimensional form next. $\lambda = \frac{a}{h}$

$$q_1 = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx\, dy = \frac{K}{H}\iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\, dr\, ds$$

$$= \frac{n}{mR}\left[4a_{i,2}\,a_{j,2} + 4\left(a_{i,2}\,a_{j,5} + a_{i,4}\,a_{j,4} + a_{i,5}\,a_{j,2}\right)/3\right.$$

$$\left.+4a_{i,5}\,a_{j,5}/5 + 16a_{i,6}\,a_{j,6}/9 + 16a_{i,8}\,a_{j,8}/3\right]$$

$$q_2 = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\, dx\, dy = \frac{H}{K}\iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\, dr\, ds$$

$$= \frac{mR}{n}\left[4a_{i,3}\,a_{j,3} + 4\left(a_{i,3}\,a_{j,6} + a_{i,4}\,a_{j,4} + a_{i,6}\,a_{j,3}\right)/3\right.$$

$$\left.+4a_{i,6}\,a_{j,6}/5 + 16a_{i,5}\,a_{j,5}/9 + 16a_{i,7}\,a_{j,7}/3\right]$$

$$q_3 = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\, dx\, dy = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial s}\, dr\, ds$$

$$= 4a_{i,2}\,a_{j,3} + 4\left(a_{i,2}\,a_{j,6} + 2a_{i,8}\,a_{j,4}\right)/3 + 4\left(2a_{i,4}\,a_{j,7} + a_{i,5}\,a_{j,3}\right)/3$$

$$+4\left(a_{i,5}\,a_{j,6} + a_{i,6}\,a_{j,5}\right)/9$$

$$q_4 = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\, dx\, dy = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\, dr\, ds$$

$$= 4a_{i,3}\,a_{j,2} + 4\left(a_{i,6}\,a_{j,2} + 2a_{i,4}\,a_{j,8}\right)/3 + 4\left(2a_{i,7}\,a_{j,4} + a_{i,3}\,a_{j,5}\right)/3$$

$$+4\left(a_{i,6}\,a_{j,5} + a_{i,5}\,a_{j,6}\right)/9$$

$$q_5 = \iint N_i \frac{\partial N_j}{\partial x}\, dx\, dy = \frac{K}{2} \iint N_i \frac{\partial N_j}{\partial r}\, dr\, ds$$

$$= \frac{1}{2mR}\left[4a_{i,1}\, a_{j,2} + 4\left(a_{i,7}\, a_{j,2} + a_{i,3}\, a_{j,4} + a_{i,1}\, a_{j,5}\right)/3\right.$$

$$+4\left(2a_{i,8}\, a_{j,2} + 2a_{i,2}\, a_{j,8}\right)/3 + 4\left(a_{i,6}\, a_{j,4} + a_{i,8}\, a_{j,5} + a_{i,4}\, a_{j,6} + 2a_{i,5}\, a_{j,8}\right)/9$$

$$\left.+4a_{i,7}\, a_{j,5}/5\right]$$

$$q_6 = \iint N_i \frac{\partial N_j}{\partial y}\, dx\, dy = \frac{H}{2} \iint N_i \frac{\partial N_j}{\partial s}\, dr\, ds$$

$$= \frac{1}{2n}\left[4a_{i,1}\, a_{j,3} + 4\left(a_{i,8}\, a_{j,3} + a_{i,2}\, a_{j,4} + a_{i,1}\, a_{j,6}\right)/3\right.$$

$$+4\left(2a_{i,7}\, a_{j,3} + 2a_{i,3}\, a_{j,7}\right)/3 + 4\left(a_{i,5}\, a_{j,4} + a_{i,7}\, a_{j,6} + 2a_{i,4}\, a_{j,5} + 2a_{i,6}\, a_{j,7}\right)/9$$

$$\left.+4a_{i,8}\, a_{j,6}/5\right]$$

$$q_7 = \iint \frac{\partial N_i}{\partial x} N_j\, dx\, dy = \frac{K}{2} \iint \frac{\partial N_i}{\partial r} N_j\, dr\, ds$$

$$= \frac{1}{2mR}\left[4a_{i,2}\, a_{j,1} + 4\left(a_{i,2}\, a_{j,7} + a_{i,4}\, a_{j,3} + a_{i,5}\, a_{j,1}\right)/3\right.$$

$$+4\left(2a_{i,2}\, a_{j,8} + 2a_{i,8}\, a_{j,2}\right)/3 + 4\left(a_{i,4}\, a_{j,6} + a_{i,5}\, a_{j,8} + 2a_{i,6}\, a_{j,4} + 2a_{i,8}\, a_{j,5}\right)/9$$

$$\left.+4a_{i,5}\, a_{j,7}/5\right]$$

$$q_8 = \iint \frac{\partial N_i}{\partial y} N_j\, dx\, dy = \frac{H}{2} \iint \frac{\partial N_i}{\partial r} N_j\, dr\, ds$$

$$= \frac{1}{2n}\left[4a_{i,3}\, a_{j,1} + 4\left(a_{i,3}\, a_{j,8} + a_{i,4}\, a_{j,2} + a_{i,6}\, a_{j,1}\right)/3\right.$$

$$+4\left(a_{i,3}\, a_{j,7} + 2a_{i,7}\, a_{j,3}\right)/3 + 4\left(a_{i,4}\, a_{j,5} + a_{i,6}\, a_{j,7} + 2a_{i,5}\, a_{j,4} + 2a_{i,7}\, a_{j,6}\right)/9$$

$$\left.+4a_{i,6}\, a_{j,8}/5\right]$$

$$q_9 = \iint N_i N_j\, dx\, dy = \frac{HK}{2} \iint N_i N_j\, dr\, ds$$

$$= \frac{1}{2nmR}\left[4a_{i,1}\, a_{j,1} + 4\left(a_{i,1}\, a_{j,7} + a_{i,3}\, a_{j,3} + a_{i,7}\, a_{j,1}\right)/3\right.$$

$$+4\left(a_{i,1}\, a_{j,8} + a_{i,2}\, a_{j,2} + a_{i,8}\, a_{j,1}\right)/3 + 4\left(a_{i,5}\, a_{j,5} + a_{i,6}\, a_{j,6}\right)/15$$

$$+4\left(a_{i,1}\, a_{j,5} + a_{i,3}\, a_{j,6} + a_{i,4}\, a_{j,4} + a_{i,5}\, a_{j,2} + a_{i,6}\, a_{j,3} + a_{i,8}\, a_{j,7} + a_{i,7}\, a_{j,8}\right)/9$$

$$\left.+4\left(a_{i,7}\, a_{j,7} + a_{i,8}\, a_{j,8}\right)/5\right]$$

C.2 The element mass matrix:

The element mass matrix is a 40×40 symmetric matrix. In what follows $i,j = 0,1,2,\ldots,7$

$M_{i+5i,1+5j} = I_1 q_9/\lambda^2, \quad M_{1+5i,2+5j} = M_{1+5i,3+5j} = 0,$

$M_{1+5i,4+5j} = I_2 q_9/\lambda^2, \quad M_{1+5i,5+5j} = 0,$

$M_{2+5i,1+5j} = 0, \quad M_{2+5i,2+5j} = I_1 q_9/\lambda^2,$

$M_{2+5i,3+5j} = M_{2+5i,4+5j} = 0, \quad M_{2+5i,5+5j} = I_2 q_9/\lambda^2,$

$M_{3+5i,1+5j} = M_{3+5i,2+5j} = 0, \quad M_{3+5i,3+5j} = I_1 q_9/\lambda^2,$

$M_{3+5i,4+5j} = M_{3+5i,5+5j} = 0,$

$M_{4+5i,1+5j} = I_2 q_9/\lambda^2, \quad M_{4+5i,2+5j} = M_{4+5i,3+5j} = 0,$

$M_{4+5i,4+5j} = I_3 q_9/\lambda^2, \quad M_{4+5i,5+5j} = 0,$

$M_{5+5i,1+5j} = 0, \quad M_{5+5i,2+5j} = I_2 q_9/\lambda^2,$

$M_{5+5i,3+5j} = M_{5+5i,4+5j} = 0, \quad M_{5+5i,5+5j} = I_3 q_9/\lambda^2,$

In the above expressions $\lambda = \frac{a}{h}, q_9$ as given earlier, and

$$[I_1, I_2, I_3] = \int_{-h/2}^{h/2} \rho[1, z, z^2] dz$$

Appendix (D)

The non – dimensional quantities used in the analysis are:

$$\bar{u} = \left(\frac{a}{h^2}\right) u, \quad \bar{v} = \left(\frac{a}{h^2}\right) v, \quad \bar{w} = \left(\frac{1}{h}\right) w,$$

$$\bar{\phi} = \left(\frac{a}{h}\right) \phi, \quad \bar{\psi} = \left(\frac{a}{h}\right) \psi, \quad \bar{A}_{ij} = \left(\frac{1}{E_1 h}\right) A_{ij}$$

$$\bar{B}_{ij} = \left(\frac{1}{E_1 h^2}\right) B_{ij}, \quad \bar{D}_{ij} = \left(\frac{1}{E_1 h^3}\right) A_{ij}, \quad (i,j = 1,2,6)$$

$$\bar{A}_{ij} = \left(\frac{1}{E_1 h}\right) A_{ij} (i,j = 4,5), \quad \bar{b} = \frac{b}{a}$$

$$\bar{I}_1 = \left(\frac{1}{\rho h}\right) I, \quad \bar{I}_2 = \left(\frac{1}{\rho h^2}\right) I_2, \quad \bar{I}_3 = \left(\frac{1}{\rho h^3}\right) I_3$$

$$\bar{\omega} = \omega a^2 \sqrt{\frac{\rho}{E_1 h^2}} \ , \ \frac{a}{b} = R.$$

Where E_1 and ρ are values of the modulus of elasticity in the fiber direction and density respectively.

Appendix (E)

Table (E.1): The lower five non – dimensional frequency for 0/90/0 square free plate (FF), Material (1).

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	1.2479	2.0251	2.8961	4.2066	4.4498
10	1.3838	2.2570	3.4476	5.5152	5.7187
20	1.4484	2.3343	3.7039	6.1144	6.2747
40	1.4778	2.3578	3.8067	6.3033	6.4703
80	1.4892	2.3670	3.8450	6.3596	6.5530

Table (E.2): The lower five non – dimensional frequency for 0/90/0 rectangular free plate (FF), a=2b, Material (1).

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	2.1989	4.2022	5.1133	5.7835	6.2414
10	2.6309	5.5053	7.0703	8.1147	9.1478
20	2.8447	6.1009	8.1645	9.0484	10.4586
40	2.9422	6.2885	8.5953	9.3596	10.9692
80	2.9840	6.3449	8.7449	9.4545	11.1487

Table (E.3): The lower five non – dimensional frequency for 0/90/0 square built – in plate (CC), Material (1).

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	3.1557	4.6206	5.5782	6.5952	6.6979
10	4.8650	6.9972	9.5565	10.4752	10.9679
20	6.2026	8.7281	13.4390	14.0008	15.6234
40	6.8003	9.5240	14.9401	16.7723	18.5544
80	7.0110	9.8515	15.6913	17.9214	19.8190

Table (E.4): The lower five non – dimensional frequency for 0/90/0 rectangular built – in plate (CC), a=2b. Material (1).

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	5.1359	6.9117	9.1580	9.3671	10.3633
10	8.1796	11.6939	15.7229	16.6184	18.0160
20	10.5114	16.6526	22.3574	25.7329	26.3561

40	11.5481	19.6775	26.1188	31.6192	32.9856
80	11.9174	20.9315	27.6753	33.9162	36.6256

Table (E.5): The lower five non – dimensional frequency for 0/90/0 square free plate (FF), Material (2).

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	0.6257	1.2935	1.6235	2.2504	2.6139
10	0.6967	1.4975	1.9708	3.6590	4.0168
20	0.7321	1.5707	2.1331	4.1775	4.6498
40	0.7487	1.5924	2.1964	4.3625	4.8949
80	0.7552	1.6002	2.2196	4.4354	4.9890

Table (E.6): The lower five non – dimensional frequency for 0/90/0 rectangular free plate (FF), a=2b, Material (2)

a/h	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	1.1015	2.6138	2.8064	3.2414	3.4612
10	1.3227	4.2543	4.5860	5.1748	5.5556
20	1.4373	5.5473	5.9950	6.0658	6.5358
40	1.4920	6.1009	6.2987	6.7527	6.9254
80	1.5159	6.2605	6.3986	6.9756	7.0543