



# Effects of Boundary Conditions on Cross-Ply Laminated Composite Beams

Imad-Eldin Mahmoud Mahdi<sup>1</sup>, Osama Mohammed Elmardi Suleiman<sup>2</sup>  
and Ahmed F. Algarray<sup>3</sup>

Assistant Professors<sup>1,2</sup>

Department of Mechanical Engineering,  
Faculty of Engineering and Technology,  
Nile Valley University, Atbara,  
River Nile State,

Sudan

Lecturer<sup>3</sup>

Department of Mechanical Engineering,  
Faculty of Engineering,  
Red Sea University, Port Sudan,  
Red Sea State,

Sudan

---

## ABSTRACT

*In this study, the effect of the end conditions of cross-ply laminated composite beams (CLCB) on their non-dimensional natural frequencies of free vibration was investigated. The problem is analyzed and solved using the energy approach which is formulated by a finite element model. In that model, a three-noded element with three degrees of freedom at each node is assumed. Numerical results were verified by comparisons with other relevant works. The end conditions of beams are: clamped -free (CF), hinged -hinged (HH), clamped -clamped (CC), hinged -clamped (HC), hinged -free (HF), free -free (FF). Each beam has either movable ends or immovable ends. It is found that the more constrained beams have the higher values of natural frequencies of transverse vibration. However, the free-free and hinged-free beams are found to have the highest frequencies of transverse vibration amongst all beams although they look less constrained. This behavior is due to the fact that the first mode of the two beams is equal zero (rigid body motion), and replaced by the second mode to be the fundamental mode. The values of the natural frequencies of longitudinal modes are found to be the same for all beams with movable ends since they are generated by longitudinal movements only. But for immovable ends, the clamped-free and hinged-free beams have equal frequencies in longitudinal vibration, and those of the other beams are also the same.*

**Keywords:** Finite Element Method, End Conditions, Cross Ply Laminates, First Order Shear Deformation, Natural Frequencies.

---

## 1. INTRODUCTION

Cross-ply laminated composite beams (CLCB) are those having alternating layers of material bonded together in some manner, and found in many of the products used in our day-to-day lives, like cars, boats, machines. Additionally, they are used in many critical industrial, aerospace and military applications. The benefits of composites have fueled growth of new applications in

markets such as transportation, construction, corrosion-resistance, marine, infrastructure, consumer products, electrical, aircraft and business equipment's[1], [2], [3], [4], [5], [6] and [7].

It is very important to estimate the values of natural frequencies of CLCB in order to avoid failure by avoiding resonance when these beams subjected to dynamic loads. Usually the non-dimensionalized frequencies are computed and can then be applied for any beam size. Many parameters are affecting the values of natural frequency; one of them is the end conditions. The first-order shear deformation theory was used by Teboub and Hajela [8] to analyze the free vibration of generally layered composite beams. Banerjee J.R. [9] applied an exact solution to predict the frequency equations and mode shapes of free vibration of laminated composite beams. Also, Abramovich and Livshits [10] presented exact solutions for the free vibration of non-symmetrically laminated cross-ply composite beams. A beam finite element based on layer wise trigonometric shear deformation theory is presented by Shimpi R.P., and Ainapure A.V. [11]. A free vibration analysis of fiber-reinforced composite beams was carried out by Marur S.R. and Kant T. [12]. The authors applied the higher-order theories and finite element method to predict the values of natural frequencies of laminated composite beams.

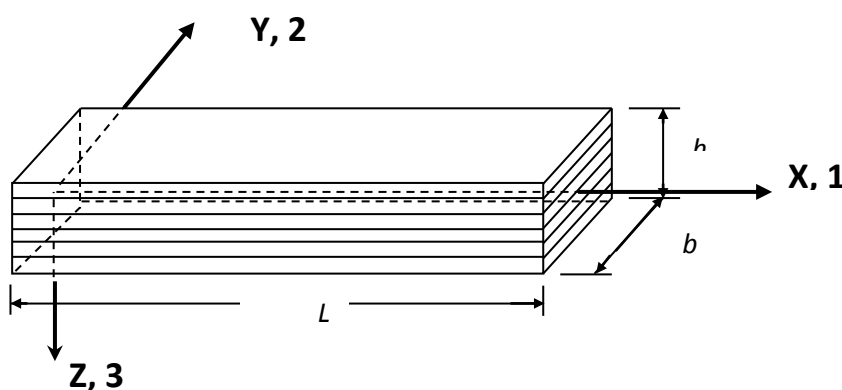
## 2. MATERIAL AND GEOMETRY

AS/3501-6 graphite-epoxy material was used for all numerical results because of its wide applications in modern industries. The mechanical properties of this material are tabulated in Table 1.

**Table 1** Mechanical Properties of AS/3501-6 graphite-epoxy material

Property	Magnitude
E1	145 GN/m <sup>2</sup>
E2	9.6 GN/m <sup>2</sup>
G12	4.1 GN/m <sup>2</sup>
G13	4.1 GN/m <sup>2</sup>
G23	3.4 GN/m <sup>2</sup>
Poisson's ratio ( $\nu$ )	0.3
Density ( $\rho$ )	1520 kg/m <sup>3</sup>

Figure 1 below shows the geometry of a beam drawn in the three dimensions X, Y, and Z or 1, 2, and 3.



**Figure 1** The geometry of a laminated composite beam.

## 3. MATHEMATICAL FORMULATIONS

The time-dependent axial and transverse displacements fields are:

$$\left. \begin{aligned} U(x, z, t) &= u(x, t) + z\phi(x, t) \\ W(x, z, t) &= w(x, t) \end{aligned} \right\} \quad (1)$$

Where,  $u$  and  $w$  are the axial and transverse displacements at the mid-plane,  $z$  is the perpendicular distance from the mid-plane to the layer plane,  $\phi$  is the rotation of a plane after deformation, and  $t$  is the time. The strain- displacement relations are:

$$\left. \begin{aligned} \varepsilon_1 &= \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} + z \frac{\partial \phi}{\partial x} \\ \varepsilon_5 &= \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = \frac{\partial w}{\partial x} + \phi \end{aligned} \right\} \quad (2)$$

Where the subscripts have the same meanings as those used in 3-D elasticity formulation, i.e.  $\varepsilon_1$  is the axial or longitudinal strain, and  $\varepsilon_5$  is the through-thickness shear strain. The stress-strain relationship of a lamina can be shown as:

$$\{\sigma_i\} = [\bar{C}_{ij}] \{\varepsilon_i\} \quad (3)$$

$$\text{Where, } \left. \begin{aligned} \{\sigma\}^T &= \{\sigma_1 \quad \sigma_5\} \\ [\bar{C}_{ij}] &= \begin{bmatrix} \bar{C}_{11} & 0 \\ 0 & \bar{C}_{55} \end{bmatrix} \\ \{\varepsilon\}^t &= [\varepsilon_1 \quad \varepsilon_5] \end{aligned} \right\} \quad (4)$$

The elastic constants  $[\bar{C}_{11}]$  and  $[\bar{C}_{55}]$  for orthotropic beams can be expressed as:

$$\left. \begin{aligned} \bar{C}_{11} &= C_{11} \cdot \cos^4 \theta + C_{22} \cdot \sin^4 \theta + 2(C_{12} + 2C_{66}) \cdot \sin^2 \theta \cdot \cos^2 \theta \\ \bar{C}_{55} &= C_{44} \cdot \sin^2 \theta + C_{55} \cdot \cos^2 \theta \end{aligned} \right\} \quad (5)$$

$$\text{Where; } C_{66} = G_{12}; C_{44} = G_{23}; C_{55} = G_{13}; C_{12} = \nu_{12} C_{22} = \nu_{21} C_{11} \quad (6)$$

#### 4. FINITE ELEMENT MODELING

Applying the energy approach for the beam element shown in Figure 2, the strain energy stored is given by:

$$U_s = \frac{1}{2} \int_e \{\varepsilon\}^T \{\sigma\} dV \quad (7)$$

Where,  $dV = bdx dz$  and the subscript, e, means one element

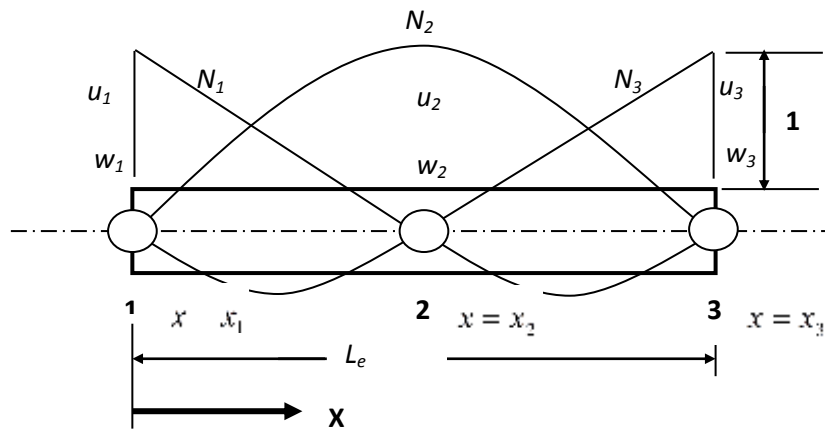


Figure 2, A 3-noded finite element with nine degrees of freedom and shape functions.

Also, the kinetic energy is found as follows;

$$K.E. = \frac{1}{2} b\rho \int_e \left[ \frac{\partial^2 W}{\partial t^2} W + \frac{\partial^2 U}{\partial t^2} U \right] dx dz \quad (8)$$

The degrees of freedom at each node are; the axial displacement  $u$ , deflection  $w$ , and rotation  $\phi$ , and can be written in terms of their nodal values as follows:

$$[u, w, \phi] = \sum_{i=1}^{i=3} [N_i u_i, N_i w_i, N_i \phi_i] \quad (9)$$

Where,  $N_i$  is shape function and assumed as a second-order polynomial as:

$$N_i = a_i + b_i x + c_i x^2 \quad (i = 1,2,3) \quad (10)$$

The constants  $a_i$ ,  $b_i$  and  $c_i$  can be computed for each element from the following data:

$$N_i = \begin{cases} 1 & \text{at } x = x_i \quad (i = 1,2,3) \\ 0 & \text{at } x \neq x_i \end{cases} \quad (11)$$

Equations (7), and (8) leads to the final form of the non-dimensional element stiffness and inertia matrices  $[K]^e$ , and  $[M]^e$  respectively. These individual element stiffness and inertia matrices must be linked together or assembled to form the global matrices and to characterize the unified behavior of the entire beam. Therefore, the global stiffness and inertia matrices are given respectively by,

$$\left. \begin{aligned} [K] &= \sum_{n=1}^N [K]^e \\ [M] &= \sum_{n=1}^N [M]^e \end{aligned} \right\} \quad (12)$$

Where,  $N$  is the total number of beam elements.

The beam end conditions which considered are; (1) Clamped-free beam (CF), (2) Hinged-hinged beam (HH), (3) Clamped-clamped beam (CC), (4) Hinged-clamped beam (HC), (5) Hinged-free beam (HF), and (6) Free-free beam (FF). Each beam has either movable ends or immovable ends. In the former group, the axial displacement at both beam-ends is considered, while neglected for the latter group.

The solution can be obtained after the incorporation of ends conditions which will modify both stiffness and inertia matrices. Thus, the non-dimensional natural frequencies can be determined from the relation:

$$[M]^{-1}[K] - \omega^2 I = 0 \quad (13)$$

Where, I is an identity matrix, and  $\omega$  is the non-dimensionalized natural frequencies, which can be computed by computing the square root of the eigenvalues of the matrix  $[M]^{-1}[K]$  using a suitable computer program (Here MATLAB was used).

## 5. METHOD VALIDITY

In order to check the validity of the present method, some comparisons were performed. These comparisons were selected to cover the cases of symmetrically and non-symmetrically laminated beams. Table 2 presents a comparison with Abramovich and Livshits [10] of the non-dimensional frequencies of symmetric [0/90/90/0] cross-ply graphite-epoxy beams for aspect ratio of (L/h = 10). The end conditions considered are hinged-hinged, fixed-free, and fixed-fixed with immovable ends. Here, the authors introduced the secondary effect of coupling between bending and torsion in their analysis, which is small, compared with the other secondary effects. For the hinged-hinged beam, a percentage difference of less than 0.16% was recorded for the fundamental frequency, and less than 0.54% for both fixed-free and fixed-fixed beams. This difference was observed to increase as the mode order increases (less than 1.4%) for the seventh mode for all beams considered.

**Table 2 Non-dimensional frequencies  $[\bar{\omega} = \omega.L^2 \sqrt{\rho/E_1 h^2}]$  of symmetric [0/90/90/0] cross-ply beams (L/h = 10)**

Mode No.	Hinged-hinged		Fixed-free		Fixed-fixed	
	Present	Ref. [10]	Present	Ref. [10]	Present	Ref. [10]
1	2.3157	2.3194	0.8866	0.8819	3.6855	3.7576
2	6.9813	7.0029	4.1062	4.0259	7.7244	7.8718
3	12.004	12.037	8.9536	9.1085	12.381	12.573
4	17.010	17.015	11.504*	12.193*	17.192	17.373
5	22.015	21.907	13.924	14.080	22.119	22.200
6	23.007*	23.007*	18.980	18.980	23.007*	23.007*
7	27.094	27.094	24.037	24.037	27.125	27.125

(\*) Modes with predominance of longitudinal vibration.

Another comparison was carried out with the results of Marur and Kant [12]. Table 3 compares the first six modes of the non-dimensionalized frequencies  $[\bar{\omega} = \omega.L^2 \sqrt{\rho/E_1 h^2}]$  of symmetric [0/90/90/0] cross-ply, graphite-epoxy, clamped-free beam with aspect ratio of (L/h = 15).

**Table 3 Non-dimensional natural frequencies  $[\bar{\omega} = \omega.L^2 \sqrt{\rho/E_1 h^2}]$  of symmetric [0/90/90/0] cross-ply clamped-free beam. (L/h = 15).**

Mode No.	Present	Ref. [12]
1	0.9238	0.924
2	4.8886	4.985
3	11.4556	11.832
4	17.2550*	-
5	18.8481	19.573

6	26.7793	27.720
---	---------	--------

(\*) Modes with predominance of longitudinal vibration.

The authors applied the higher-order shear deformation theory (HOSDT) in the analysis, whereas, the present method uses first-order shear deformation theory. The comparison shows a percentage difference of less than 0.03% for the fundamental mode of vibration. This difference increases with the mode order to less than 3.6% for the sixth mode.

## 6. NUMERICAL RESULTS AND DISCUSSIONS

Table 4 shows the first ten modes of free vibration of CLCB with immovable ends (i.e. axial movement is restricted), while table 5 shows those for movable ends. Generally, it is found that more constrained beams have high values of natural frequencies. However, the free-free and hinged-free beams are found to have the highest frequencies amongst all beams although they look less constrained. This behavior is due the fact that the first mode of the two beams is equal zero and replaced by the second mode. The fundamental mode shapes of both beams are straight lines, Figure 3, and this due to the rigid motion in this mode where there is no vibrating motion.

**Table 4** Non-dimensional natural frequencies  $\left[\bar{\omega} = \omega \sqrt{\rho L^4 / E_1 h^2}\right]$  of a symmetric cross-ply [90/-90/90/90] laminated beam with immovable ends, (L/h = 10).

Mode No.	Beam type					
	CF	HH	CC	HC	HF	FF
1	0.2597	0.7224	1.5544	1.1020	1.1184	1.6077
2	1.5522	2.7525	3.9654	3.3519	3.4324	4.1694
3	4.0539*	5.7774	7.1377	6.4646	4.0539*	7.5785
4	4.0677	8.1077*	8.1077*	8.1077*	6.6580	8.1077*
5	7.3503	9.4753	10.8006	10.1515	10.4868	11.5233
6	11.1503	13.5995	14.7896	14.2085	12.1616*	15.7969
7	12.1616*	16.2155*	16.2155*	16.2155*	14.6913	16.2155*
8	15.2783	17.9847	18.9972	18.5027	19.1205	20.2606
9	19.6134	22.5259	23.3543	22.9488	20.2694*	24.3234*
10	20.2694*	24.3234*	24.3234*	24.3234*	23.6776	24.8245

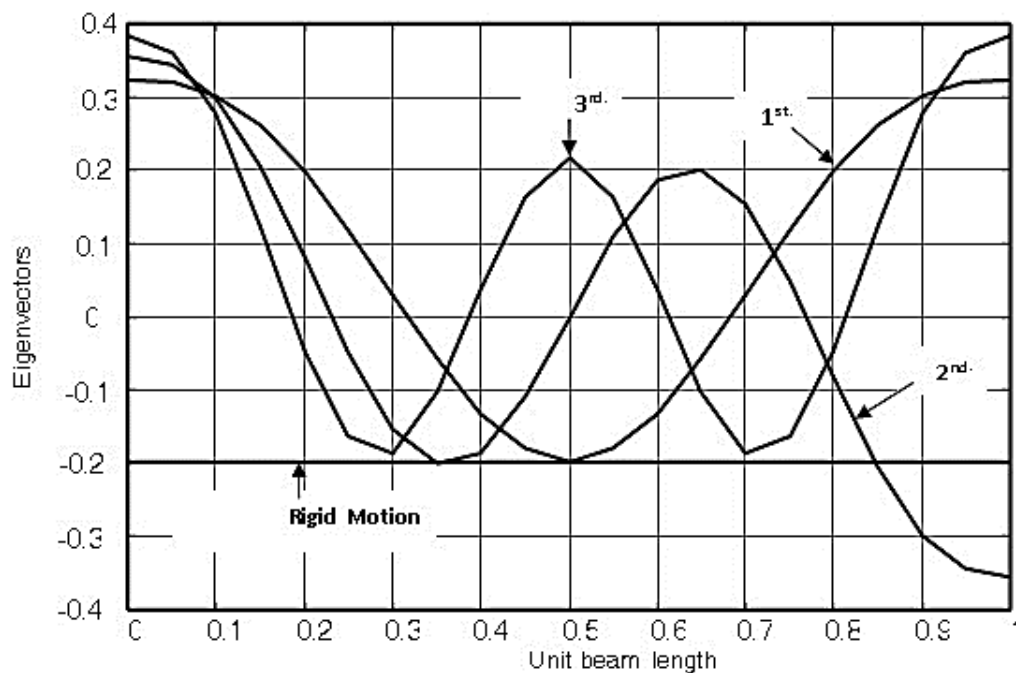
(\*) Modes with predominance of longitudinal vibration.

Also, it can be observed from the results in tables 4 and 5 that the values of the non-dimensionalized natural frequencies of the transverse modes are not affected by the longitudinal movements of the ends since these modes are generated by lateral movements only. However, the values of the natural frequencies of longitudinal modes are found to be the same for all beams with movable ends since they are generated by longitudinal movements only. It could be noticed that the values of non-dimensionalized natural frequencies of the longitudinal vibration for the clamped-free (CF) and hinged-free (HF) beams are equal, and those of the other beams are also the same. This phenomenon occurs since both clamped-free and hinged-free beams with immovable ends are the same when restricted from executing longitudinal motion at the ends. Similarly, the rest of beams with immovable ends have the same longitudinal end conditions.

**Table 5 Non-dimensional natural frequencies  $\left[\bar{\omega} = \omega\sqrt{\rho L^4 / E_1 h^2}\right]$  of a symmetric cross-ply [90/-90/-90/90] laminated beam with movable ends, (L/h = 10).**

Mode No.	CF	HH	CC	HC	HF	FF
1	0.2597	0.7224	1.5544	1.1020	1.1184	1.6077
2	1.5522	2.7525	3.9654	3.3519	3.4324	4.1694
3	4.0677	5.7774	7.1377	6.4646	6.6580	7.5785
4	7.3503	8.1077*	8.1077*	8.1077*	8.1077*	8.1077*
5	8.1077*	9.4753	10.8006	10.1515	10.4868	11.5233
6	11.1503	13.5995	14.7896	14.2085	14.6913	15.7969
7	15.2783	16.2155*	16.2155*	16.2155*	16.2155*	16.2155*
8	16.2155*	17.9847	18.9972	18.5027	19.1205	20.2606
9	19.6134	22.5259	23.3543	22.9488	23.6776	24.3234*
10	24.0758	24.3234*	24.3234*	24.3234*	24.3234*	24.8245

(\*) Modes with predominance of longitudinal vibration.



**Figure 3, Transverse mode shapes of a symmetric of a free-free beam, L/h=10**

### 7. CONCLUSIONS

In the present study, the effects of end conditions of CLCB on their non-dimensionalized natural frequencies are studied. The problem is modeled and analyzed by the finite element method. Twelve end conditions were studied which are clamped-free, hinged-hinged, clamped-clamped, hinged-clamped, hinged-free, and free-free beams with immovable and movable ends. It is found that more constrained beams have high values of natural frequencies. However, the free-free and hinged-free beams are found to have the highest frequencies amongst all beams although they look less constrained. The transverse modes are not affected by the longitudinal movements of the ends since these modes are generated by lateral movements only. The values of the natural frequencies of longitudinal modes are found to be the same for all beams with movable ends since they are generated by longitudinal movements only. But for immovable ends, the clamped-free and hinged-free beams have equal frequencies in longitudinal vibration, and those of the other beams are also the same.

## ACKNOWLEDGEMENT

The authors would like to acknowledge with deep thanks and profound gratitude Mr. Osama Mahmoud of Daniya Center for Publishing and Printing Services, Atbara, who spent many hours in editing, re – editing of the manuscript in compliance with the standard format of International Journal of Engineering Research and Advanced Technology (IJERAT).

## REFERENCES

1. R.A. Jafari-Talookolaei and M.T.Ahmadian ' Free Vibration Analysis of a Cross-Ply Laminated Composite Beam on Pasternak Foundation', Journal of Computer Science 3 (1): pp. (51-56), (2007).
2. Mahmoud Yassin Osman and Osama Mohammed Elmardi Suleiman, 'Large deflection of composite beams', International Journal of Engineering Research and Advanced Technology(IJERAT), Vol.03, Issue 3, (2017), pp. (26 – 39).
3. Osama Mohammed Elmardi Suleiman, 'Literature review on imperfection of composite laminated plates', Journal of Microscopy and Ultrastructure, Vol.03, 5 (2017), pp. (119 – 122).
4. Mahmoud Yassin Osman and Osama Mohammed Elmardi Suleiman, 'Free vibration of laminated plates', International Journal of Engineering Research and Advanced Technology (IJERAT), Vol.03, Issue 4, (2017), pp. (31 – 47).
5. Mahmoud Yassin Osman and Osama Mohammed Elmardi Suleiman, 'Free vibration analysis of laminated composite beams using finite element method', International Journal of Engineering Research and Advanced Technology (IJERAT), Vol.03, Issue 2, (2017), pp. (5 – 22).
6. Mahmoud Yassin Osman and Osama Mohammed Elmardi Suleiman, 'Buckling analysis of thin laminated composite plates using finite element method', International Journal of Engineering Research and Advanced Technology (IJERAT), Vol.03, Issue 3, (2017), pp. (1 – 18).
7. Osama Mohammed Elmardi Suleiman, 'Verification of dynamic relaxation (DR) method in isotropic, orthotropic and laminated plates using small deflection theory', International Journal of Advanced Science and Technology (IJAST), Vol. 72 (2017), pp. (37 –48).
8. Teboub Y. and Hajela P. ' Free vibration of generally layered composite beams using symbolic computations', Composite Structure 33, pp. (123-134), (1995).
9. Banerjee J.R. 'Explicit analytical expression for frequency equation and mode shapes of composite beams', International Journal of Solid and Structures 38, pp. (2415-2426), (2001).
10. Abramovich H. and Livshits A. ' Free vibrations of non-symmetric cross-ply laminated composite beams', Journal of sound and vibration 176(5), pp. (597-612), (1994).
11. Shimpi R.P. and Ainapure A.V. ' A beam finite element based on layer wise trigonometric shear deformation theory', Composite Structures 53, pp. (153-162), (2001).
12. Marur S.R. and Kant T. 'Free vibration analysis of fiber-reinforced composite beams using higher-order theories and finite element modeling', Journal of sound and vibration 194, pp. (337-351), (1996).