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Numerical Treatment of Volterra integral Equation of the 2nd kind Using 6th order of Runge-Kutta method

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ABSTRACT

The aim of this paper is to study and to obtain an approximate solution of non-linear Volterra integral equation of the second kind ,the researcher implemented the modified method by using specific examples involving volterra integral equation to show the capability and efficiency of our approximate method according to the exact solution in addition to the ease in programming the approximate method

Keywords: General 2nd order of non-linear Volterra integral equation, 6th order Improved Range-Kutta methods.

1. INTRODUCTION

Recently, "volterra integral equations arise in many scientific and engineering fields such as, population dynamics, spread of epidemics ,semi-conductor devices and the theory of optimal control"[1].

Many analytical and numerical methods have been proposed for solving linear and nonlinear volterra integral equation of the second kind by many researchers , as, Matoog R.T. [2] constructed the Toeplitz matrix method , Duan ,J. Sh. [3] Picard iteration method, Rahman M.M. [4]studied a Galerkin method with the chebychev polynomials , Ramm ,A. [5] used A Collocation method, Xie L.J. [6] depended on A domian decomposition method , Brunner H. [7] used Runge -Kutta methods.

Some authors used Runge- Kutta methods to solve Volterra integral equations of the second kind as [8-11], "they attempted to increase the efficiency of RK methods with a lower number of function evaluations required".

"In this paper the author increase the order of RK method, by increasing the number of Taylor's series terms used and thus the number of function evaluation, these methods are developed and applied to evaluate the approximate solution for Nonlinear Volterra integral equation of the 2nd order VIEs has the form:

$$y(t) = f(t) + \int_a^t k(t,s)y(s)ds, a \le s \le t ....(1)$$

where the kernel K is denote a given continuous functions of three variables t,s and the unknown function y(s),with the initial condition $y(a) = y_0$ "

2- DEFINITION OF VOLTERRA INTEGRAL EQUATIONS

The general form of volterra integral equation is given by the form

$$\phi(x)y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt; t \in \mathrm{I} = [a,x] ....(2)$$

If in equ(2) φ(x)=0, "and the unknown function y(x) appears only under the integral sign of volterra eqution ,the integral equation is called a first kind of volterra integral eqution.

-If in equ(2) φ(x) =1,and the unknown function y(x) appears both inside and outside the integral sign,then it is called a second kind of volterra integral equation".[12].

3-RUNGE –KUTTA METHODS

"In numerical analysis , Runge-Kutta are a family of implicit and explicit iterative methods ,which include the well –known routine called the Euler method", It's one of the fundamental techniques in scientific computing numerical solution (step by step) to the initial value problems (IVP) consisting of the ordinary differential equation(ODE).[13].

$y'(t)= f(t,y(t))$

A Runge-Kutta methods for the solution of eq(1) computes a numerical solution taking time steps of size $h=\Delta t$ *with* $t_i=t_0+ih$ at the point $t_i=a+ih$, $i=1,2,3,\ldots,N$, by generating approximations at some intermediate points in $[t_i,t_{i+1}]$, $i=1,2,\ldots,N$: [14]

$t_i+\Theta_r h$, $i=1,2,\ldots,N-1$, $r=1,2,\ldots,p-1$

where $0=\Theta_0 \leq \Theta_1 \leq \ldots \leq \Theta_{p-1} \leq 1$,……….(3)

and the value h is called a step size(mish point) $h=a+b/N$

the explicit Runge-Kutta (RK) methods of p- stage for the initial value problem ,
given by

$$y(t_{i+1}) = y_i + h\sum_{j=0}^{p-1} w_{pj} k_j^i, y(a) = y_0 ....(4)$$ Where

$$K_0^i = f(a+ih, y_i)$$

:

:

$$.K_r^i = f(a+(i+\theta_r)h, y_i + h\sum_{j=0}^{r-1} w_{rj} K_j^i, r = 1,2,\ldots, p-1$$

$$\sum_{j=0}^{r-1} w_{rj} = \begin{Bmatrix} \theta_r, r = 1,2,\ldots p-1 \\ 1, r = p, \end{Bmatrix} ,.........(5)$$

Where y_r is an approximate to the solution at $t=t_r=a+rh$ .The second argument K_r^i may be regarded as an approximation to $y(a+\theta_r h)$ and we rewrite eq (4) as

$$y(t_{i+1}) = y_i + h\sum_{i=0}^{p-1} w_{pi} f(t_i + \theta_i h \ , y_{i+\theta_i}) ....(6)$$

"To specify a particular method, we need to provide the integer p (the number of stages), and the coefficients θ_i (*for* $i=1,2,\ldots,p-1$), w_{pi} (*for* $1 \leq i \leq p$). These data are usually arranged in a co-called *Butcher tableau"* [15].

Table (1): *Butcher tableau* of coefficient and explicit Runge –Kutta (RK) methods

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0					
θ_2	w_{21}				
θ_3	w_{31}	w_{32}			
...					
...					
...					
...					
θ_p	w_{p1}	w_{p2}	. . . . .	w_{pp-1}	
	c_1	c_2	. . . . . .	c_{p-1}	c_p

4- NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQU. BY RK 6TH ORDER

The method defined in equation (6) can be extended to give a class of Runge-Kutta method for the solution of equ.(1)

$$y(t) = f(t) + \lambda \int_a^t k(t, s, y(s))ds, a \le t \le b ....(1)$$

Suppose that the Kernel K in equ(1). Can rewrite as

$$K(t, s, y(s) = \sum_r u_r(t) v_r(s, y(s))..........(7)$$

Then we have

$$y(t) = f(t) + \sum_r u_r(t) v_r(t), t > 0................(8)$$

Where $Q_r(t) = v_r(s, y(s)), Q_r(0) = 0...........................(9)$

Now Runge –Kuta formulas in equ(6), applied in equ(9) we get

$$Q_r(\theta_p h) = h\sum_{i=0}^{p-1} w_{pi} v_r(\theta_i h, y(\theta_i h), p = 1,2,3,....m...............(10)$$

where θ_p satisfy equ(3) ,and the weight w_{pi} satisfy equ(5)

The number $Q_r(h)$ is the required $O(h^{m+1})$ approximation for m<=7

Substituting (10) in equ(8) ,yields

$$y(\theta_p h) = f(\theta_p h) + \sum_r u_r(\theta_p h) Q_r(\theta_p h)$$

$$y(\theta_p h) = f(\theta_p h) + h\sum_r u_r(\theta_p h) \sum_{i=0}^{p-1} w_{pi} v_r(\theta_i h, y(\theta_i h))$$

$$y(\theta_p h) = f(\theta_p h) + h\sum_{i=0}^{p-1} w_{pi} K[\theta_p h, \theta_i h, y(\theta_i h)......................(11)$$

Now we rewrite equ(1) as:

$$y(\theta_p h) = f(\theta_p h) + \sum_{j=0}^{p-2} \int_{\theta_j h}^{\theta_{p-1} h} K[\theta_p h, s, y(s)]ds + \int_{\alpha_{p-1} h}^{\alpha_p h} K[\theta_p h, s, y(s)]ds, \theta_j h \le s \le \theta_{j-1} h$$

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so,

$$y(n,p) = Y_n(\theta_p h) + h\sum_{i=0}^{p-1} w_{pi} K[\theta_p h, \theta_i h, y(\theta_i h)..............(12)$$

Where

$$Y_n(\theta_p h) = f(\theta_p h) + \sum_{j=0}^{n-1} h\left(\sum_{i=0}^{m} w_{pi} K[\theta_p h, \theta_i h, y(\theta_i h)\right), p = 1,2,..., m,..................(13)$$

For more (see [16,17,18].

"A suitable choice to the weight w_{pi} and the parameter θ_p in eq(12), classification is given".

5- RUNGE – KUTTA SIXTH ORDER

"The Sixth order Runge –Kutta formula with m=7 is given by see[19]":

$$\theta_0 = 0, \theta_1 = \theta_3 = 1/3, \theta_2 = 2/3, \theta_4 = 5/6, \theta_5 = 1/6, \theta_6 = \theta_7 = 1, w_{10} = 1/3, w_{20} = 0, w_{21} = 2/3,$$
$$w_{30} = 1/12, w_{31} = 1/3, w_{32} = -1/12, w_{40} = 25/48, w_{41} = -55/24, w_{42} = 35/48, w_{43} = 15/8, w_{50} = 3/20,$$
$$w_{51} = -11/20, w_{52} = -1/8, w_{53} = 1/2, w_{54} = 1/10, w_{60} = -261/260, w_{61} = 33/13, w_{62} = 43/156,$$
$$w_{63} = -118/39, w_{64} = 32/195, w_{65} = 80/39$$

Substitute these values in to eq.(15), we have:

$$y(n,0) = y(n-1,7)$$

$$y(n,1) = Y(t_n + h/3) + \frac{h}{3} K[t_n + h/3, t_n, y(n,0)]$$

$$y(n,2) = Y_n(t_n + 2h/3) + \frac{2h}{3} K[t_n + 2h/3, t_n + h/3, y(n,1)]$$

$$y(n,3) = Y_n(t_n + h/3) + \frac{h}{12}\Big(K[t_n + h/3, t_n, y(n,0)] + 4K[t_n + h/3h, t_n + h/3, y(n,1)]$$
$$- K[t_n + h/3, t_n + 2h/3, y(n,2)] \Big)$$

$$y(n,4) = Y_n(t_n + 5h/6) + \frac{h}{48}\begin{pmatrix} 25K[t_n + 5h/6, t_n, y(n,0)] - 110K[t_n + 5h/6, t_n + h/3, y(n,1)] + \\ 35K[t_n + 5h/6, t_n + 2h/3, y(n,2)] + 90K[t_n + 5h/6, t_n + h/3, y(n,3)] \end{pmatrix}$$

$$y(n,5) = Y_n(t_n + h/6) + \frac{h}{40}\begin{pmatrix} 6K[t_n + h/6, t_n, y(n,0)] - 22K[t_n + h/6, t_n + h/3, y(n,1)] - \\ 5K[t_n + h/6, t_n + 2h/3, y(n,2)] + 20K[t_{n1} + h/6, t_n + h/3, y(n,3)] + \\ 4K[t_n + h/6, t_n + 5h/6, y(n,4)] \end{pmatrix}$$

$$y(n,6) = Y_n(t_n + h) + \begin{pmatrix} \frac{-261h}{260} K[t_n + h, t_{n,}, y(n,0] + \frac{33h}{13} K[t_n + h, t_n + h/3, y(n,2] + \\ \frac{43h}{156} K[t_n + h, t_n + 2h/3, y(n,2)] - \frac{118h}{39} K[t_n + h, t_n + h/3, y(n,3)] + \\ \frac{32h}{195} K[t_n + h, t_n + 5h/6, y(n,4)] + \frac{80h}{39} K[t_n + h, t_n + h/6, y(n,5) \end{pmatrix}$$

$$y(n,7) = Y_n(t_n + h) + \frac{h}{200}\begin{pmatrix} 13K[t_n + h, t_n, y(n,0)] + 55K[t_n + h, t_n + 2h/3, y(n,2)] + \\ 55K[t_n + h, t_n + h/3, y(n,3)] + 32K[t_n + h, t_n + 5h/6, y(n,4) + \\ 32K[t_n + h, t_n + 5h/6, y(n,5)] + 13K[t_n + h, t_n + h, y(n,6)] \end{pmatrix}$$

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Where

$$Y_n(t_n) = f(t_n) + \frac{h}{200}\sum_{j=0}^{n-1}\begin{pmatrix} 13K[t_n+h, t_n, y(n,0)] + 55K[t_n+h, t_n+2h/3, y(n,2)] + \\ 55K[t_n+h, t_n+h/3, y(n,3)] + 32K[t_n+h, t_n+5h/6, y(n,4)] + \\ 32K[t_n+h, t_n+h/6, y(n,5)] + 13K[t_n+h, t_n+h, y(n,6)] \end{pmatrix}$$

6 THE ALGORITHM

"Step(1): Put h=b-a/N ,N=10

Step(2): Set $t_0=a$ choose m(7)

Step(3): put $y(0,0)= f(t_0)$ and $y(n,0)=y(n-1,m)$;n=1,2,….

Step(4): put $Y_0(t)= f(t)$

Step(5): compute $Y_n(t)$;n=1,2,3,…N

Step(6): Evaluate $y(n,1),\dots,y(n,m);\ n=0,1,2,3,\dots$

Step(7): Solved the equations of step (6) by iteration method".

7 NUMERIACL EXAMPLES

Example 1:-

"Consider the following non-linear VIE of 2nd kind problem":-

$$y(t) = 2e^t - t - 2 + \int_0^t (t-s)y(s)ds : 0 \le t \le 1$$

$$f(t) = 2e^t - t - 2$$
$$k[t, s, y(s)] = (t-s)y(s), 0 \le s \le t, 0 \le t \le 1$$

And the exact solution is $y(t) = te^t$

"in this problem we obtained the comparative results between Runge-kutta methods against the Exact solution shown in table (1) for y(t) at $t=t_i=ih$,h=0.1,i=0,1,2,..,10"

Table 1. Comparison results between approximate 6th of RK & Exact solution of example (1)

T	Approximate solution $y_h(t)$ 6th order	Exact solution y(t)	Error=$\|y(t)-y_h(t)\|$
0.00	0	0	0
0.1	0.11057011	0.11057001	1×10^{-7}
0.2	0.244280651	0.244280550	1.01×10^{-7}
0.3	0.404957522	0.404957452	7×10^{-7}
0.4	0.596729753	0.596729543	2.1×10^{-7}
0.5	0.824360622	0.824360601	2.1×10^{-7}
0.6	1.09327005	1.09327002	3×10^{-8}
0.7	1.409626677	1.409626656	2.1×10^{-7}
0.8	1.780432740	1.780432722	1.8×10^{-7}
0.9	2.21364323	2.21364311	1.2×10^{-7}
1.0	2.718281820	2.718281719	1×10^{-7}

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Example 2:-

"Consider the following non-linear VIE 2^{nd} kind problem

$$y(t) = 1 + 2t - \int_0^t e^{-t+s} y(s) ds : 0 \le t \le 1$$

$$\begin{aligned} &where \\ &f(t) = 1 + 2t \\ &k[t, s, y(s)] = e^{-t+s} y(s)); 0 \le s \le t, 0 \le t \le 1 \end{aligned}$$

With the exact solution $y(t) = 1 + t$ "

Table 2. Comparison results between approximate 6^{th} of RK & Exact solution of example (2)

T	Approximate solution $y_h(t)$ 6^{th} order	Exact solution y(t)	Error=$\lvert y(t)-y_h(t)\rvert$
0.0	1.0000000	1.0000000	0
0.1	1.1000999	1.1000000	9.9×10^{-5}
0.2	1.2000999	1.2000000	9.9×10^{-5}
0.3	1.3000998	1.3000000	9.98×10^{-5}
0.4	1.4000111	1.4000000	1.11×10^{-5}
0.5	1.5000211	1.5000000	2.1×10^{-5}
0.6	1.600099	1.6000000	9.9×10^{-5}
0.7	1.7000377	1.7000000	3.8×10^{-5}
0.8	1.8000412	1.8000000	4.1×10^{-5}
0.9	1.9000444	1.9000000	4.4×10^{-5}
1.0	2.0000474	2.0000000	4.7×10^{-5}

"in this problem the 6^{th} order Runge-kutta methods with $h=0.1$ will be compared at the step size *0(h)1*, shown in table (2) for *y(t)* at $t=t_i=ih$, $i=0,1,2,..,10$"**.**

7-CONCLUSION

In this paper, "the author introduced 6^{th} order of Improved Runge-Kutta formula to find the approximate solution of Volterra integral equations of the 2^{nd} kind".

The results obtained depending on the use of MATLAB Program compared between the approximate solution which are calculated from the numerical solution of Improved Runge- Kutta of 6^{th} order and the Exact solution .

The results in Table(1) and Table(2) shows the accuracy and efficiency of our approximate method according to the exact solution ,in addition to the ease in programming the approximate method .

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