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Elastic Deflection of Beams and frames Using Dream Method

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ABSTRACT

This paper presents a new graphical method for the analysis of linear elastic structures called Dream Method. The method states that the strain energy of a linear elastic multi-member structure undergoing small deformation is the sum of moment of area of the bending moment diagrams of the individual members each about its own diagram base divided by the flexural rigidity of the member. The displacement at any point on the structure is equal to the partial derivative of the total strain energy with respect to the load at that point. The method is applied successfully in the analysis of a built-in stepped beam, a beam on three supports, and a four-member frame.

Keywords: Bending, Deflection, Dream, Energy, Graphical, Structure.

1. INTRODUCTION

There are many factors which must be considered in the analysis of engineering structures. These factors are related to geometry of the structure, material, and type of deformation. A structure can be one-dimensional like a beam, two-dimensional like a plate, and three-dimensional like a space frame. The behavior of the material under load can be linear elastic, nonlinear elastic, or inelastic. The composition of the material can be homogeneous or non-homogeneous. The dependence of the mechanical properties on direction classifies the material as isotropic, orthotropic, or anisotropic. Finally, deformation can be small or large. We add one last factor about the desired analysis which can be stress analysis, displacement analysis or stability of a structure. The classical methods of analysis assume the material is linear elastic, homogeneous, and isotropic, and the deformations are small. It is obvious that those methods suit metals and in particular steel which is the largest structural metallic material known to mankind, and which made our civilization possible.

Analysis of linear elastic structures undergoing small deformation has received a great attention for many decades in the past. Some researchers in the field of mechanics of materials may have concluded that the days of analytic treatment of engineering structures made of traditional materials have gone forever. Such a conclusion can be justified due to introduction of new materials with tailored properties, efficient numerical techniques, and fast computers that can handle the most complex structures.

The author spent several years teaching mechanics of materials to undergraduate engineering classes, and perhaps those classical methods were always at the back of his head because many students find difficulties in applying them. An attempt which made this work possible, happened at the beginning not in reality but in a dream. At first, the author referred to the method *Graphical Energy Method* because it embraces graphics and energy. Later he preferred to call it *Dream Method* to avoid confusion with other graphical methods. We shall survey literature to highlight the difference between *Dream Method* and other analytical and graphical methods which are covered by many books on mechanics of materials or strength of materials as those cited in *References*.

1.1 Moment Area Method:

The Moment Area Method uses the bending moment diagram (BMD) of a beam to determine its slope and deflection. In simple cases, the change of slope between any two sections along the beam, say x_1 - where the slope is zero - and

x_2, is equal to the area A of the segment of the BMD between the two sections divided by the flexural rigidity EI where E is modulus of elasticity and I is second moment of area of the beam cross-section. The deflection at x_2 is equal to the moment of the area A about section x_2, again divided by EI. The method is involving and unsuitable for indeterminate beams and frames.

2. CONJUGATE BEAM METHOD

The Conjugate Beam Method involves creating a *conjugate beam* that is loaded with the BMD of the real beam divided by the flexural rigidity EI. The conditions at the supports are modified so that a free end in the real beam is replaced by a fixed end in the conjugate beam, and vice-verse, whereas a simple support in the real beam remains as it is in the conjugate beam. The slope and deflection of the real beam are determined from the shear force diagram (SFD) and BMD. The shear force and bending moment at any point on the conjugate beam are equal to the slope and deflection at the corresponding point on the real beam, respectively. The method can be applied to determinate and indeterminate beams. However, constructing and analyzing the conjugate beam can be complex especially for indeterminate beams under multiple loads. It should be noted that, the method is not based on a mathematical formulation.

3. DOUBLE INTEGRATION METHOD

The Double Integration Method is a technique used to determine slope and deflection of beams under various loading conditions. It is based on solving the differential equation of the elastic curve of a beam which relates the bending moment $M(x)$ to the curvature of the beam. The mathematical form is

$$EI\frac{d^2v}{dx^2} = M(x)$$

Where v is the deflection at position x along the beam. When the beam is subjected to multiple loads, the BMD consists of multiple segments. The number of equations to be solved is equal to the number of segments which is very challenging. The complication is overcome using *Macaulay's Method*. By introducing singularity terms, one single bending moment equation for the entire beam is obtained. The two constants, due to integration, are determined from conditions at the supports. The method is applicable to a wide range of loading as well as to determinate and indeterminate beams. However, the method is unsuitable for non-prismatic beams and frames, and computation may become laborious for beams with complex loading and support conditions.

4. STRAIN ENERGY METHODS:

Energy, being a scalar quantity, is easier to work with than to work with vector quantities. The energy methods are based on the principle of conservation of energy where work done by external loads is stored as strain energy in the structure due to deformation. Energy methods are widely used in the analysis of various types of structures under multiple loads. Here is a short account of two energy methods.

(a) Virtual Work Method:

The Virtual Work Method (sometimes called *Fictitious Load Method*) applies a fictitious or dummy load to the beam at the point where deflection or slope is to be evaluated. The load is typically a unit load. The work done by the fictitious load moving through the deflection caused by the real load is equated to the internal energy acquired by the beam due to both the real and fictitious loads. The deflection is computed using integral equation which involves moments as follows:

$$v = \int \frac{1}{EI} M_a(x) M_f(x) dx$$

Where v is deflection, $M_a(x)$ is bending moment due to the actual load at position x, and $M_f(x)$ is bending moment due to fictitious unit load. When slope is desired, the applied load is a unit moment. It is clear that the method requires integration which can be complicated. In addition to that, the method does not lend itself to frames.

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(b) Castigliano's Method:

Castigliano's Method is used in the analysis of various elastic structures under multiple loads. The theorem relates the displacement at any point in the structure to the partial derivative of the total strain energy with respect to (wrt) the applied load at the point. The theorem can be expressed mathematically as follows:

$$v_i = \frac{\partial U}{\partial P_i}$$

Where v_i and P_i are the displacement and load at point i, respectively, and U is the total strain energy of the structure.

Dream Method

Figure 1 shows segment i of a multi-segment BMD of a loaded beam. A_i is the area of the segment bounded by x_1 and x_2.

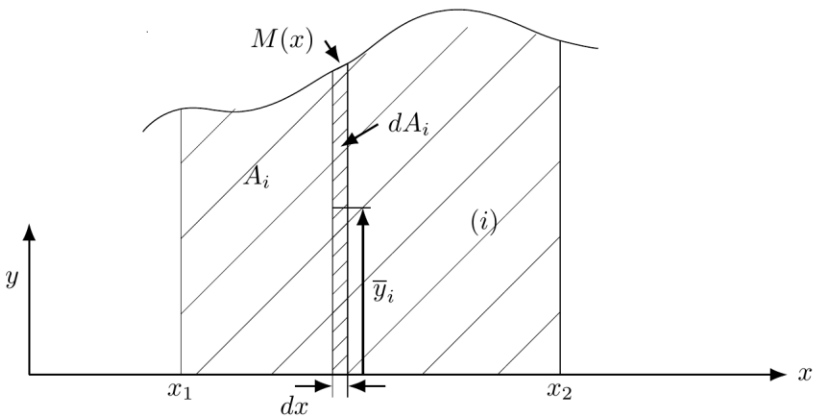

Figure 1: Multi-segment BMD of a beam

The rectangular shaded element is $M(x)$ in height and dx in width, and its area is $dA_i = M(x)dx$. The centroid of the element is at $y_i = \frac{1}{2}M(x)$ from the base of the diagram. The moment of area of the rectangular element about the base of the diagram is

$$dA_i y_i = \frac{1}{2}M^2(x)dx$$

The moment of area of segment, $A_i\overline{y}_i$, about the base of the diagram is

$$A_i\overline{y}_i = \int_{x_1}^{x_2} \frac{1}{2}M^2(x)dx$$

According to *Dream Method*, the strain energy in segment, U_i, is

$$U_i = \frac{A_i\overline{y}_i}{(EI)_i}$$

Where $(EI)_i$ is flexural rigidity of the beam along the segment.
The total strain energy of the beam is the sum of strain energy of the individual segments.

$$U = \sum_1^n U_i = \sum_1^n \frac{A_i\overline{y}_i}{(EI)_i}$$

Where n is the number of segments.
When a beam or a frame is acted on by a single load, P, the displacement at the load, v, can be obtained by equating

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the the total strain energy to the work done by the load $U = \frac{1}{2}Pv$. When the beam is acted on by multiple loads, the displacement, v_k, at load P_k is obtained from the partial derivative of the strain energy wrt P_k

$$v_k = \frac{\partial U}{\partial P_k}$$

It should be noted that the term *displacement* refers to linear and angular displacements, and *load* refers to a force or a concentrated bending moment, and the displacement is always in direction of the load. Several determinate and indeterminate structures have been treated including beams and frames, subjected to various types of loads. Analyses are carried using symbols or numerical data when that is available. Only three of the undertaken analyses are included in this paper, and these are:

1. A built-in stepped beam under a point load.
2. A beam on three supports under a point load.
3. A four-member frame under a point load.

The objectives of the mentioned analyses are:

1. To demonstrate *Dream Method* procedure.
2. To highlight any complications, if found, associated with application of the method.
3. To facilitate comparison between *Dream Method* and other methods and in particular Castigliano's Method which shares some common features with *Dream Method*.
4. To determine the advantages of using numerical data rather than working with symbols.

5. APPLICATIONS OF DREAM METHOD

Application (1)

Consider the built-in stepped beam of length l shown in Figure 2(a). The flexural rigidity of the middle portion is twice as much as that of the ends. The beam is subjected to a point load P at the middle. It is desired to find deflection under the load.

Solution:

The BMD is shown in Figure 2(b).

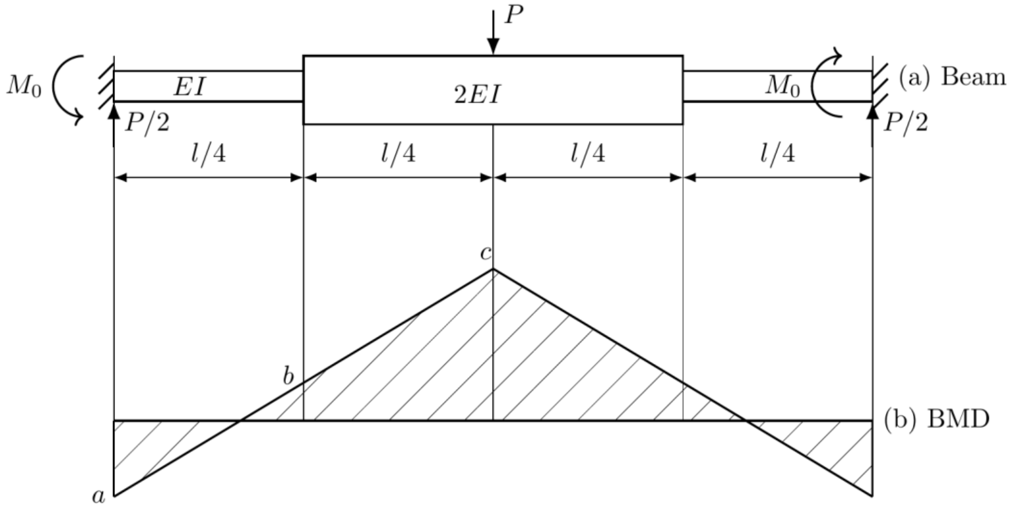

Figure 2: A built-in stepped beam under a point load

The principal values of the bending moment a, b, and c are as follows:

$$a = -M_0, \quad b = -M_0 + \frac{Pl}{8}, \quad c = -M_0 + \frac{Pl}{4}$$

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Let x be an axis along the beam with origin at the left end. Let the area of the bending moment in the range $0 \le x \le l/4$ be A_1, and the distance of its centroid from the base of the diagram be $\overline{y}_1$, then

$$A_1 = \frac{l}{8}(a+b), \quad \overline{y}_1 = \frac{a^2+b^2+ab}{3(a+b)}$$

The strain energy is

$$U_1 = \frac{A_1\overline{y}_1}{EI} = \frac{l}{24EI}(a^2+b^2+ab)$$

Let the area of the bending moment in the range $l/4 \le x \le l/2$ be A_2, and the distance of its centroid from the base of the diagram be $\overline{y}_2$, then

$$A_2 = \frac{l}{8}(b+c), \quad \overline{y}_2 = \frac{b^2+c^2+bc}{3(b+c)}$$

The strain energy is

$$U_2 = \frac{A_2\overline{y}_2}{2EI} = \frac{l}{48EI}(b^2+c^2+bc)$$

Due to symmetry, the total strain energy of the beam is

$$U = 2(U_1+U_2) \tag{1}$$

$$= \frac{l}{24EI}(2a^2+3b^2+c^2+2ab+bc)$$

The slope of the deflection at the ends is the partial derivative of the total stain energy wrt M_0, that is

$$\phi = \frac{\partial U}{\partial M_0} = \frac{l}{24EI}\left[(4a+2b)\frac{\partial a}{\partial M_0} + (2a+6b+c)\frac{\partial b}{\partial M_0} + (b+2c)\frac{\partial c}{\partial M_0}\right]$$

Substitute for a, b, c, and their derivatives wrt M_0, and set $\phi = 0$ to get

$$M_0 = \frac{5PL}{48}$$

Now we proceed to find deflection at the middle of the beam by partially differentiating the total strain energy given by equation (1) wrt to P, i.e.

$$v = \frac{\partial U}{\partial P} = \frac{l}{24EI}\left[(4a+2b)\frac{\partial a}{\partial P} + (2a+6b+c)\frac{\partial b}{\partial P} + (b+2c)\frac{\partial c}{\partial P}\right]$$

Substitute for $a = -5Pl/48,\ b = Pl/48,\ c = 7Pl/48$, and their derivatives wrt P to get:

$$v = \frac{11PL^3}{3072EI}$$

Application (2)

Consider a beam on three supports of length l, acted on by a point load at a distance $l/4$ from the left support as shown in Figure 3(a). It is desired to find deflection under load.

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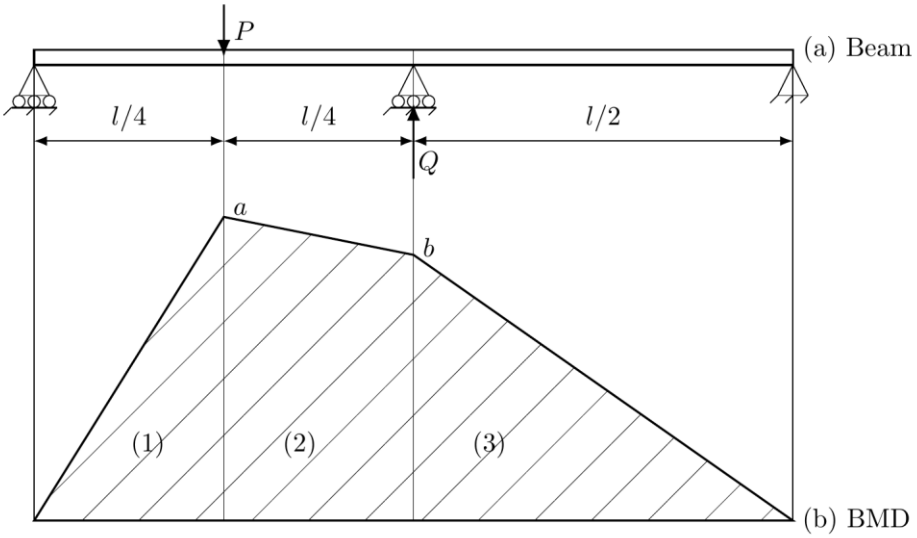

Figure 3: A beam on three supports subjected to a point load

Solution:

To determine the deflection where the load is applied, we need to find the reaction at the middle support, say Q. The BMD is shown in figure 3(b). The principal values of the bending moment are:

$$a = \frac{3Pl}{16} - \frac{Ql}{8}, \quad b = \frac{Pl}{8} - \frac{Ql}{4}$$

The BMD is divided into three segments denoted by (1), (2), and (3). The areas of the segments are A_1, A_2, and A_3, respectively. The distances of the centroids from base of diagram are $\bar{y}_1, \bar{y}_2$, and $\bar{y}_3$, respectively. These parameters are as follows:

$$A_1 = \frac{l}{8}a, \quad \bar{y}_1 = \frac{a}{3} \quad A_2 = \frac{l}{8}(a+b), \quad \bar{y}_2 = \frac{a^2+b^2+ab}{3(a+b)}, \quad A_3 = \frac{l}{4}b, \quad \bar{y}_3 = \frac{b}{3}$$

The moment of area of BMD about base of diagram, $A\bar{y}$, is the sum of moments of area of the three segments. i.e.

$$A\bar{y} = \sum_{1}^{3} A_i\,\bar{y}_i = \frac{l}{24}(2a^2 + 3b^2 + ab))$$

The total strain energy is

$$U = \frac{A\bar{y}}{EI} = \frac{l}{24EI}(2a^2 + 3b^2 + ab) \qquad (2)$$

The deflection at the middle support, v_Q, is obtained from the partial derivative of the strain energy wrt Q i.e.

$$v_Q = \frac{\partial U}{\partial Q} = \frac{l}{24EI}\left[(4a+b)\frac{\partial a}{\partial Q} + (a+6b)\frac{\partial b}{\partial Q}\right]$$

Substitute for a, and b, and their derivatives wrt Q, and set $v_Q = 0$ to get

$$Q = \frac{11P}{16}$$

Now we proceed to find deflection under the applied load, v_P, from the partial derivative of equation (2) wrt P.

$$v_P = \frac{\partial U}{\partial P} = \frac{l}{24EI}\left[(4a+b)\frac{\partial a}{\partial P} + (a+6b)\frac{\partial b}{\partial P}\right]$$

The new values of a, and b are as follows:

$$a = \frac{13Pl}{128}, \quad b = -\frac{3pl}{64}$$

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Substitute for a, and b, and their derivatives wrt to P to get the deflection under the load:

$$v_P = \frac{23Pl^3}{12288EI}$$

Application (3)

Consider a rectangular-shaped frame, of height l and length $2l$ loaded at the center of the vertical member, A, with a load, P, as shown in Figure 4(a). It is desired to find deflection under load given the flexural rigidity EI.

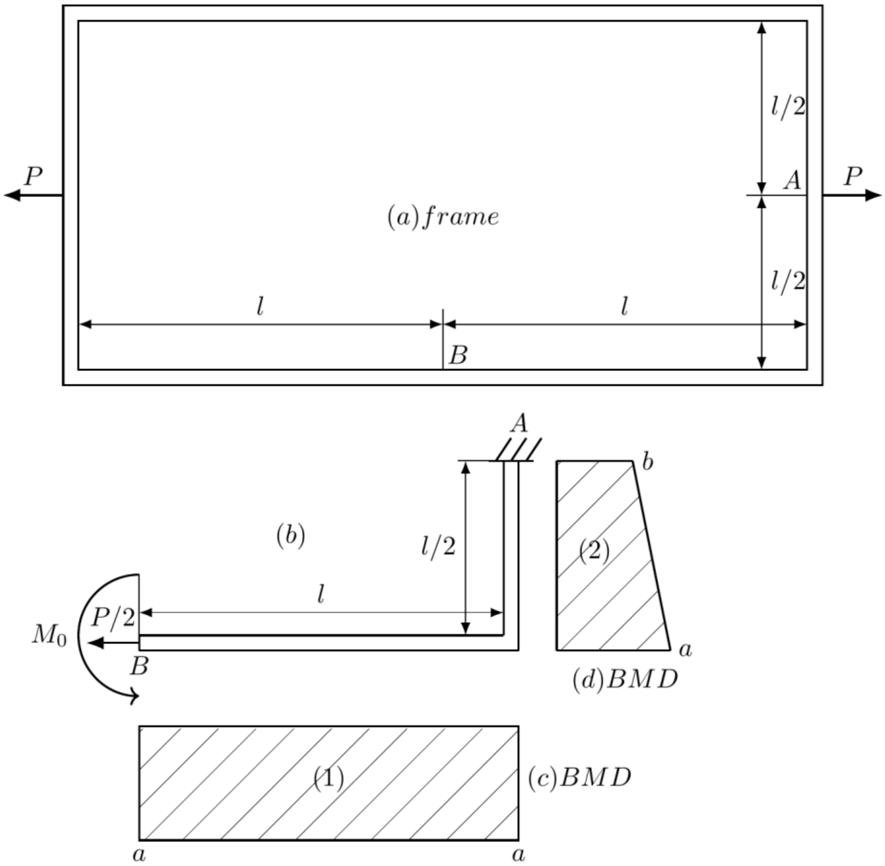

Figure 4: A four-member frame with a point load

Solution:

Due to symmetry, we consider one quarter of the frame as shown in Figure 4(b). The reactions at B are $P/2$ and bending moment M_0. To find deflection where the load is applied, we have to express M_0 in terms of P. The bending moment diagrams of the horizontal and vertical members of the frame are shown in Figure 4(c) & 4(d), denoted by (1) and (2), respectively. The principal bending moment values are:

$$a = M_0, \quad b = M_0 - \frac{Pl}{4}$$

Areas of the bending moment diagrams of the horizontal and vertical members are A_1 and A_2, respectively. The distances of the centroid of A_1 and A_2 from the base of the relevant BMD are $\bar{y}_1$ and $\bar{y}_2$, respectively. These parameters are as follows:

$$A_1 = la, \quad \bar{y}_1 = \frac{a}{2}, \quad A_2 = \frac{l}{4}(a+b), \quad \bar{y}_2 = \frac{a^2+b^2+ab}{3(a+b)}$$

Moment of area of bending moment diagram for the entire frame, $A\bar{y}$, is sum of moment of area of bending moment diagrams of both members.

$$A\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2 = \frac{l}{12}(7a^2+b^2+ab)$$

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Total strain energy of the frame is four times the strain energy of one quarter.

$$U = 4\frac{A\bar{y}}{EI} = \frac{1}{3EI}(7a^2 + b^2 + ab) \tag{3}$$

The rotation, ϕ, at point B is obtained from the partial derivative of the strain energy wrt M_0.

$$\phi = \frac{\partial U}{\partial M_0} = \frac{1}{3EI}\left[(14a + b)\frac{\partial a}{\partial M_0} + (a + 2b)\frac{\partial b}{\partial M_0}\right]$$

Substitute for a and b and their derivatives wrt M_0 and set $\phi = 0$ to get:

$$M_0 = \frac{Pl}{24}$$

Now we proceed to find deflection where the load is applied by finding the partial derivative of the total strain energy, given by equation (3), wrt P

$$v = \frac{\partial U}{\partial P} = \frac{1}{3EI}\left[(14a + b)\frac{\partial a}{\partial P} + (a + 2b)\frac{\partial b}{\partial P}\right]$$

The new values of $a = PL/24$ and $b = -5Pl/24$. Substitute these parameters and their derivatives wrt P to get

$$v = \frac{Pl^3}{32EI}$$

6. DISCUSSION

Dream Method, presented in this paper, is a graphical technique in which strain energy is expressed in terms of moment of area of BMD of a structure about base of the diagram, and its flexural rigidity. The displacement at any point is obtained by equating the strain energy to the work done by the load acting at the point in simple cases or from the partial derivative of the strain energy wrt the load in complex cases. When there is no point load where slope or deflection is evaluated, a fictitious load is applied. Up to the author knowledge this technique has never been used before.

Dream Method and Castigliano's Method have some common features. Both methods are based on strain energy acquired by a structure due to deformation caused by external loads. Castigliano's Method approach is analytical whereas *Dream Method* approach is graphical. Castigliano's method involves integration whereas *Dream Method* involves sketching BMD and computing its area and centroid. Each approach has its own merits. When a beam or a frame is under multiple point loads, the BMD consists of triangles and trapeziums. The areas of these shapes can be computed and their centroids located with almost no difficulty. When analyzing simple structures such as a cantilever beam or a simply supported beam under a single load, the pencil and paper work required by *Dream Method* is considerably less than that required by other methods. We intend to discuss the complications that an analyst may face during application of *Dream Method*. We shall also compare these complications with those associated with two widely used methods: Castigliano's Method and Double Integration Method. We consider two sources of complications that may be encountered in the analysis of beams and frames subjected to complex loading. The first source is related to multiple loads or multi-segment BMD. The second source is related to working with symbols. We shall discuss in detail these two sources of complication and suggest means of overcoming them.

(a) Complication due to multiple loads:

The bending moment changes along a structure every time load changes that leads to multi-segment BMD. Hence, as the number of loads increases, the number of segments increases. The total strain energy of such a structure is sum of the energy of its segments. So more segments means more computations. *Dream Method* and Castigliano's Method are affected to the same extent. The Double Integration Method is affected as well but to a less degree, thanks to *Macaulay's* method which through introduction of singularity terms, the bending moment for the entire beam is a single expression irrespective of the number segments. It should be re-iterated that the Double Integration Method is unsuitable for the analysis of non-prismatic beams and frames. To reduce the degree of complexity associated with

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multi-segment BMD, the analyst can seek help of the superposition technique. For example an equation derived to compute displacement for an individual load can be used to compute displacements due to some other loads.

(b) Complication due to working with symbols:

A few structures - none included in this paper - have been analyzed using numerical data rather than symbols. It was found that working with numerical values reduces the amount of paper-and-pencil work. That is because when using numerical data, some calculations are carried out as the analysis progresses while when using symbols a built-up continues till the end of the analysis. Hence, the analyst can use numerical data whenever that is available.

7. CONCLUSIONS

1. *Dream Method* theorem is stated, proved, and verified.
2. *Dream Method* is applied successfully to determine displacement of a built-in stepped beam, a beam on three supports, and a four-member frame under point loads.
3. Pencil and paper work required by *Dream Method* to determine displacement of a simple structure under a point load is less than that required by other methods.
4. *Dream Method* can be applied to beams and frames under a single point load without complication.
5. The *Method of Superposition* can be used to reduce the amount of computation when a beam or a frame is under multiple loads.
6. Analysis of structures using numerical data rather than symbols reduces the amount of computation.

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